ON STOCHASTIC FRACTIONAL VOLterra EQUATIONS IN HILBERT SPACE

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Abstract. In this paper, stochastic Volterra equations, particularly fractional, in Hilbert space are studied. Sufficient conditions for existence of strong solutions are provided.

1. Introduction. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) be a stochastic basis and \(H\) a separable Hilbert space. In this paper we consider the stochastic Volterra equations in \(H\) of the form

\[
X(t) = X(0) + \int_0^t a(t - \tau)AX(\tau)d\tau + \int_0^t \Psi(\tau)dW(\tau), \quad t \geq 0.
\]

In (1), \(X(0)\) is an \(H\)-valued \(\mathcal{F}_0\)-measurable random variable and \(a \in L^1_{\text{loc}}(\mathbb{R}^+)\) is a scalar kernel. The operator \(A\) is closed linear unbounded in \(H\) with a dense domain \(D(A)\) equipped with the graph norm \(| \cdot |_{D(A)}\), i.e. \(|h|_{D(A)} := (|h|^2_H + |Ah|^2_H)^{1/2}\), where \(| \cdot |_H\) denotes the norm in \(H\). \(W\) is a cylindrical Wiener process (see e.g. [3] or [7] for the definition, properties and the stochastic integral with respect to that process) on another separable Hilbert space \(U\), with the covariance operator \(Q \in L(U)\). \(Q\) is a linear symmetric positive operator with \(\text{Tr} Q = +\infty\) and \(\Psi\) is an appropriate process defined below.

Equations (1) contain important special cases, e.g. heat, wave and integro-differential equations. Moreover, (1) are motivated by a wide class of model problems and correspond to abstract stochastic versions of several deterministic problems, mentioned, e.g. in [13] (see also the references therein).

In order to provide a sense for the integral \(\int_0^t \Psi(\tau)dW(\tau)\), the process \(\Psi(t)\), \(t \geq 0\), has to be an operator-valued process (see, e.g. [14]). We define the subspace \(U_0 := Q^{1/2}(U)\) of the space \(U\) endowed with the inner product \(\langle u, v \rangle_{U_0} := \langle Q^{-1/2}u, Q^{-1/2}v \rangle_U\). By \(L^0_2 := L_2(U_0, H)\) we denote the set of all Hilbert-Schmidt operators acting from \(U_0\) into \(H\); the set \(L^0_2\) equipped with the norm \(||C||_{L_2(U_0,H)} := \left( \sum_{k=1}^{+\infty} |Cu_k|^2_H \right)^{1/2} \), is a separable Hilbert space.

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By $\mathcal{N}^2(0, T; L^2_0)$, where $T < +\infty$ is fixed, we denote a Hilbert space of all $L^2_0$-predictable processes $\Psi$ such that $\|\Psi\|_T < +\infty$, where

$$
\|\Psi\|_T := \left\{ E \left( \int_0^T \|\Psi(\tau)\|_{L^2_0}^2 \, d\tau \right) \right\}^{\frac{1}{2}} = \left\{ E \int_0^T \left[ \text{Tr}(\Psi(\tau)Q^2)(\Psi(\tau)Q^2)^* \right] \, d\tau \right\}^{\frac{1}{2}}.
$$

If $\Psi \in \mathcal{N}^2(0, T; L^2_0)$, then the integral $\int_0^T \Psi(\tau) \, dW(\tau)$ makes sense.

Let us note that the results obtained below for cylindrical Wiener process (Tr $Q = +\infty$) hold for genuine Wiener process (Tr $Q < +\infty$), too. In the latter case, that is, if $Q$ is a nuclear operator, $L(U, H) \subset L_2(U_0, H)$ and then the stochastic integral $\int_0^T \Psi(\tau) \, dW(\tau)$ is well defined (for details, see [13]).

In this paper, we use the so-called resolvent approach to the Volterra equation (1) (for details we refer to [13]).

First, we recall some definitions connected with deterministic version of (1), that is, the equation

$$
u(t) = \int_0^t a(t - \tau) A \nu(\tau) \, d\tau + f(t), \quad t \geq 0,
$$

(2)

where $f$ is an $H$-valued function. In (2), the kernel function $a(t)$ and the operator $A$ are the same like previously.

**Definition 1.** A family $(S(t))_{t \geq 0}$ of bounded linear operators in $H$ is called resolvent for (2) if the following conditions are satisfied:

1. $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;
2. $S(t)$ commutes with the operator $A$;
3. the following resolvent equation holds

$$
S(t)x = x + \int_0^t a(t - \tau) A S(\tau)x \, d\tau
$$

(3)

for all $x \in D(A), \ t \geq 0$.

We will assume in the sequel that the resolvent family $(S(t))_{t \geq 0}$ to (2) exists.

Let us emphasize that the family $(S(t))_{t \geq 0}$ does not create in general any semigroup and that $S(t), \ t \geq 0$, are generated by the pair $(A, a(t))$, that is, the operator $A$ and the kernel function $a(t), \ t \geq 0$.

A consequence of the strong continuity of $S(t)$ is that $\sup_{t \leq T} \|S(t)\| < +\infty$ for any $T \geq 0$.

**Definition 2.** We say that the function $a \in L^1(0, T)$ is completely positive on $[0, T]$, if for any $\mu \geq 0$, the solutions of the equations

$$
s(t) + \mu (a * s)(t) = 1 \quad \text{and} \quad r(t) + \mu (a * r)(t) = a(t)
$$

(4)

satisfy $s(t) \geq 0$ and $r(t) \geq 0$ on $[0, T]$.

The class of completely positive kernels, introduced in [2], arise naturally in applications, see [13]. For instance, the functions $a(t) \equiv 1$, $a(t) = t$, $a(t) = e^{-t}$, $t \geq 0$, are completely positive.

**Definition 3.** Suppose $S(t), \ t \geq 0$, is a resolvent. $S(t)$ is called exponentially bounded if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$
\|S(t)\| \leq M e^{\omega t}, \quad \text{for all} \ t \geq 0 ;
$$


Let us note that contrary to $C_0$-semigroups, not every resolvent family needs to be exponentially bounded; for counterexamples we refer to [4].

In the paper, the key role is played by the following, not yet published, result providing a convergence of resolvents.

**Theorem 1.** Let $A$ be the generator of a $C_0$-semigroup in $H$ and suppose the kernel function $a$ is completely positive. Then $(A, a)$ admits an exponentially bounded resolvent $S(t)$. Moreover, there exist bounded operators $A_n$ such that $(A_n, a)$ admit resolvent families $S_n(t)$ satisfying $\|S_n(t)\| \leq Me^{w_0 t}$ ($M \geq 1$, $w_0 \geq 0$) for all $t \geq 0$, $n \in \mathbb{N}$, and

$$S_n(t)x \to S(t)x \quad \text{as} \quad n \to +\infty \quad (5)$$

for all $x \in H$, $t \geq 0$.

Additionally, the convergence is uniform in $t$ on every compact subset of $\mathbb{R}_+$.

**Proof.** The first assertion follows directly from [12, Theorem 5] (see also [13, Theorem 4.2]). Since $A$ generates a $C_0$-semigroup $T(t)$, $t \geq 0$, the resolvent set $\rho(A)$ contains the ray $[w, \infty)$ and

$$\|R(\lambda, A)^k\| \leq \frac{M}{(\lambda - w)^k} \quad \text{for} \quad \lambda > w, \quad k \in \mathbb{N},$$

where $R(\lambda, A) = (A - \lambda I)^{-1}$, $\lambda \in \rho(A)$.

Define

$$A_n := nAR(n, A) = n^2R(n, A) - nI, \quad n > w \quad (6)$$

denote the *Yosida approximation* of $A$, where $R(n, A) = (nI - A)^{-1}$. For details, see e.g. [11].

Then

$$\|e^{tA_n}\| = e^{-nt}\|e^{n^2R(n, A)t}\| \leq e^{-nt}\sum_{k=0}^{\infty} \frac{n^{2k}t^k}{k!} \|R(n, A)^k\|$$

$$\leq Me^{(n + \frac{n^2}{\lambda - w})t} = Me^{\frac{n\lambda}{\lambda - w}}.$$ 

Hence, for $n > 2w$ we obtain

$$\|e^{A_n t}\| \leq Me^{2w t}. \quad (7)$$

Taking into account the above estimate and the complete positivity of the kernel function $a$, we can follow the same steps as in [12, Theorem 5] to obtain that there exist constants $M_1 > 0$ and $w_1 \in \mathbb{R}$ (independent of $n$, due to (7)) such that

$$\|[H_n(\lambda)]^{(k)}\| \leq \frac{M_1}{(\lambda - w_1)^{k+1}} \quad \text{for} \quad \lambda > w_1,$$

where $H_n(\lambda) := (\lambda - \lambda \hat{a}(\lambda)A_n)^{-1}$. Here and in the sequel the hat indicates the Laplace transform. Hence, the generation theorem for resolvent families implies that for each $n > 2w$, the pair $(A_n, a)$ admits resolvent family $S_n(t)$ such that

$$\|S_n(t)\| \leq M_1 e^{w_1 t}. \quad (8)$$

In particular, the Laplace transform $\hat{S}_n(\lambda)$ exists and satisfies

$$\hat{S}_n(\lambda) = H_n(\lambda) = \int_0^{\infty} e^{-\lambda t}S_n(t)dt, \quad \lambda > w_1.$$
Now recall from semigroup theory that for all \( \mu \) sufficiently large we have

\[
R(\mu, A_n) = \int_0^\infty e^{-\mu t} e^{A_n t} \, dt
\]

as well as,

\[
R(\mu, A) = \int_0^\infty e^{-\mu t} T(t) \, dt.
\]

Since \( \hat{a}(\lambda) \to 0 \) as \( \lambda \to \infty \), we deduce that for all \( \lambda \) sufficiently large, we have

\[
H_n(\lambda) := \frac{1}{\lambda \hat{a}(\lambda)} R(\frac{1}{\hat{a}(\lambda)}, A_n) = \frac{1}{\lambda \hat{a}(\lambda)} \int_0^\infty e^{-1/\lambda(\hat{a}(\lambda)) t} e^{A_n t} \, dt,
\]

and

\[
H(\lambda) := \frac{1}{\lambda \hat{a}(\lambda)} R(\frac{1}{\hat{a}(\lambda)}, A) = \frac{1}{\lambda \hat{a}(\lambda)} \int_0^\infty e^{-1/\lambda(\hat{a}(\lambda)) t} T(t) \, dt.
\]

Hence, from the identity

\[
H_n(\lambda) - H(\lambda) = \frac{1}{\lambda \hat{a}(\lambda)} [R(\frac{1}{\hat{a}(\lambda)}, A_n) - R(\frac{1}{\hat{a}(\lambda)}, A)]
\]

and the fact that \( R(\mu, A_n) \to R(\mu, A) \) as \( n \to \infty \) for all \( \mu \) sufficiently large (see, e.g. [11, Lemma 7.3]), we obtain that

\[
H_n(\lambda) \to H(\lambda) \quad \text{as} \quad n \to \infty .
\]

Finally, due to (8) and (9) we can use the Trotter-Kato theorem for resolvent families of operators (cf. [9, Theorem 2.1]) and the conclusion follows. \( \square \)

**Remark 1.** (a) The convergence (5) is an extension of the result due to Clément and Nohel [2].

(b) The above theorem gives a partial answer to the following open problem for a resolvent family \( S(t) \) generated by a pair \((A, a)\): do there exist bounded linear operators \( A_n \) generating resolvent families \( S_n(t) \) such that \( S_n(t)x \to S(t)x \). In particular case \( a(t) \equiv 1, A_n \) are provided by the Hille-Yosida approximation of \( A \) and additionally \( S_n(t) = e^{tA_n} \).

2. **Probabilistic results.** In the sequel we shall use the following Probability Assumptions, abbr. (PA):

1. \( X(0) \) is an \( H \)-valued, \( \mathcal{F}_0 \)-measurable random variable;
2. \( \Psi \in N^2(0, T; L^2) \) and the interval \([0, T]\) is fixed.

The following types of definitions of solutions to (1) are possible, see [8].

**Definition 4.** Assume that (PA) hold. An \( H \)-valued predictable process \( X(t), t \in [0, T] \), is said to be a strong solution to (1), if \( X \) has a version such that \( P(X(t) \in D(A)) = 1 \) for almost all \( t \in [0, T] \); for any \( t \in [0, T] \)

\[
\int_0^t |a(t - \tau)AX(\tau)|_H \, d\tau < +\infty, \quad P-a.s.
\]

and for any \( t \in [0, T] \) the equation (1) holds \( P \)-a.s.

Let \( A^* \) be the adjoint of \( A \) with a dense domain \( D(A^*) \subset H \) and the graph norm \( | \cdot |_{D(A^*)} \) defined as follows: \( |h|_{D(A^*)} := (|h|_H^2 + |A^*h|_H^2)^{1/2} \) for \( h \in D(A^*) \).
Definition 5. Let (PA) hold. An \( H \)-valued predictable process \( X(t), \ t \in [0,T] \), is said to be a \textbf{weak solution} to (1), if \( P(\int_0^T |a(t-\tau)X(\tau)|_H d\tau < +\infty) = 1 \) and if for all \( \xi \in D(A^*) \) and all \( t \in [0,T] \) the following equation holds
\[
\langle X(t), \xi \rangle_H = \langle X(0), \xi \rangle_H + \left( \int_0^t a(t-\tau)X(\tau) d\tau, A^*\xi \right)_H \\
+ \left( \int_0^t \Psi(\tau)dW(\tau), \xi \right)_H, \ P-a.s.
\]

Definition 6. Assume that \( X(0) \) is \( \mathcal{F}_0 \)-measurable random variable. An \( H \)-valued predictable process \( X(t), \ t \in [0,T] \), is said to be a \textbf{mild solution} to the stochastic Volterra equation (1), if \( E\left(\int_0^T |S(t-\tau)\Psi(\tau)|^2_{L^2} d\tau\right) < +\infty \) for \( t \leq T \) and, for arbitrary \( t \in [0,T] \),
\[
X(t) = S(t)X(0) + \int_0^t S(t-\tau)\Psi(\tau) dW(\tau), \ P-a.s. \tag{11}
\]

The integral appearing in (11) will be called \textbf{stochastic convolution} and denoted by
\[
W^\Psi(t) := \int_0^t S(t-\tau)\Psi(\tau) dW(\tau), \quad t \geq 0, \tag{12}
\]
where \( \Psi \in \mathcal{N}^2(0;L^0_2) \).

We will show in the sequel that the convolution \( W^\Psi \) is a weak solution to (1) and next we will provide sufficient conditions under which \( W^\Psi \) is a strong solution to (1), as well.

Let us recall (from \[3\] and \[8\]) some properties of the convolution \( W^\Psi(t), t \geq 0 \).

Proposition 1. (see, e.g. \[3\] Proposition 4.15)
Assume that \( A \) is a closed linear unbounded operator with the dense domain \( D(A) \subset H \) and \( \Phi(t), \ t \in [0,T] \) is an \( L_2(U_0,H) \)-predictable process. If \( \Phi(t)(U_0) \subset D(A) \), \( P-a.s. \) for all \( t \in [0,T] \) and
\[
P \left( \int_0^T ||\Phi(s)||^2_{L^2} ds < \infty \right) = 1, \quad P \left( \int_0^T ||A\Phi(s)||^2_{L^2} ds < \infty \right) = 1,
\]
then
\[
P \left( \int_0^T \Phi(s) dW(s) \in D(A) \right) = 1
\]
and
\[
A \int_0^T \Phi(s) dW(s) = \int_0^T A\Phi(s) dW(s), \ P-a.s.
\]

For the proofs of Propositions \[2\] and \[4\] we refer to \[8\].

Proposition 2. Assume that \( \Phi \) admits resolvent operators \( S(t), t \geq 0 \). Then, for arbitrary process \( \Psi \in \mathcal{N}^2(0,T;L^0_2) \), the process \( W^\Psi(t), t \geq 0 \), given by (12) has a predictable version.

Proposition 3. Assume that \( \Psi \in \mathcal{N}^2(0,T;L^0_2) \). Then the process \( W^\Psi(t), t \geq 0 \), defined by (12) has square integrable trajectories.
Proposition 4. If $\Psi \in \mathcal{N}^2(0, T; L^2_0)$, then the stochastic convolution $W^\Psi$ fulfills the equation
\[
\langle W^\Psi(t), \xi \rangle_H = \int_0^t \langle a(t - \tau)W^\Psi(\tau), A^*\xi \rangle_H + \int_0^t \langle \xi, \Psi(\tau)dW(\tau) \rangle_H, \quad P - a.s.
\]
for any $t \in [0, T]$ and $\xi \in D(A^*)$.

Proposition 4 shows that the convolution $W^\Psi$ is a weak solution to (1) (see [8]) and enables us to formulate the following results.

Proposition 5. Let $A$ be the generator of $C_0$-semigroup in $H$ and suppose that the function $a$ is completely positive. If $\Psi$ and $A\Psi$ belong to $\mathcal{N}^2(0, T; L^2_0)$ and in addition $\Psi(t)(U_0) \subset D(A)$, $P$-a.s., then the following equality holds
\[
W^\Psi(t) = \int_0^t a(t - \tau)AW^\Psi(\tau)d\tau + \int_0^t \Psi(\tau)dW(\tau), \quad P - a.s. \tag{13}
\]

Proof. Because formula (13) holds for any bounded operator, then it holds for the Yosida approximation $A_n$ of the operator $A$, too, that is
\[
W^\Psi_n(t) = \int_0^t a(t - \tau)A_nW^\Psi_n(\tau)d\tau + \int_0^t \Psi(\tau)dW(\tau),
\]
where
\[
W^\Psi_n(t) := \int_0^t S_n(t - \tau)\Psi(\tau)dW(\tau)
\]
and
\[
A_nW^\Psi_n(t) = A_n\int_0^t S_n(t - \tau)\Psi(\tau)dW(\tau).
\]
Recall that by assumption $\Psi \in \mathcal{N}^2(0, T; L^2_0)$. Because the operators $S_n(t)$ are deterministic and bounded for any $t \in [0, T]$, $n \in \mathbb{N}$, then the operators $S_n(t - \cdot)\Psi(\cdot)$ belong to $\mathcal{N}^2(0, T; L^2_0)$, too. In consequence, the difference
\[
\Phi_n(t - \cdot) := S_n(t - \cdot)\Psi(\cdot) - S(t - \cdot)\Psi(\cdot) \tag{14}
\]
belongs to $\mathcal{N}^2(0, T; L^2_0)$ for any $t \in [0, T]$ and $n \in \mathbb{N}$. This means that
\[
\mathbb{E}\left(\int_0^t |\Phi_n(t - \tau)|_{L^2_0}^2d\tau\right) < +\infty \tag{15}
\]
for any $t \in [0, T]$.

Let us recall (see [7]) that the cylindrical Wiener process $W(t)$, $t \geq 0$, can be written in the form
\[
W(t) = \sum_{j=1}^{+\infty} f_j \beta_j(t), \tag{16}
\]
where $\{f_j\}$ is an orthonormal basis of $U_0$ and $\beta_j(t)$ are independent real Wiener processes. From (16) we have
\[
\int_0^t \Phi_n(t - \tau)dW(\tau) = \sum_{j=1}^{+\infty} \int_0^t \Phi_n(t - \tau)f_jd\beta_j(\tau). \tag{17}
\]
Then, from (15)
\[
\mathbb{E}\left[\int_0^t \left(\sum_{j=1}^{+\infty} |\Phi_n(t - \tau)f_j|_H^2\right)d\tau\right] < +\infty \tag{18}
\]
for any \( t \in [0, T] \). Next, from (17), properties of stochastic integral and (18) we obtain for any \( t \in [0, T] \), that

\[
\mathbb{E} \left| \int_0^t \Phi_n(t - \tau) \, dW(\tau) \right|^2_H = \mathbb{E} \left| \sum_{j=1}^{+\infty} \int_0^t \Phi_n(t - \tau) \, f_j \, d\beta_j(\tau) \right|^2_H \\
\leq \mathbb{E} \left[ \sum_{j=1}^{+\infty} \int_0^t |\Phi_n(t - \tau) \, f_j|^2_H \, d\tau \right] < +\infty.
\]

By Theorem 1, the convergence (5) of resolvent families is uniform in \( t \) on every compact subset of \( \mathbb{R}_+ \), particularly on the interval \([0, T]\). Then, for any fixed \( j \),

\[
\int_0^T |[S_n(T - \tau) - S(T - \tau)] \, \Psi(\tau) \, f_j|^2_H \, d\tau \longrightarrow 0, \quad \text{as} \quad n \to \infty. \tag{19}
\]

So, using (18) and (19) we can write

\[
\sup_{t \in [0, T]} \mathbb{E} \left| \int_0^t \Phi_n(t - \tau) \, dW(\tau) \right|^2_H = \sup_{t \in [0, T]} \mathbb{E} \left| \int_0^t [S_n(t - \tau) - S(t - \tau)] \, \Psi(\tau) \, dW(\tau) \right|^2_H \\
\leq \mathbb{E} \left[ \sum_{j=1}^{+\infty} \int_0^T |[S_n(T - \tau) - S(T - \tau)] \, \Psi(\tau) \, f_j|^2_H \, d\tau \right] \longrightarrow 0
\]

as \( n \to +\infty \).

Hence, by the Lebesgue dominated convergence theorem

\[
\lim_{n \to +\infty} \sup_{t \in [0, T]} \mathbb{E} \left| W_n^\Psi(t) - W^\Psi(t) \right|^2_H = 0. \tag{20}
\]

By assumption, \( \Psi(t)(U_0) \subset D(A) \), \( P \) - a.s. Because \( S(t)(D(A)) \subset D(A) \), then \( S(t - \tau)\Psi(\tau)(U_0) \subset D(A) \), \( P \) - a.s., for any \( \tau \in [0, t] \), \( t \geq 0 \). Hence, by Proposition 1, \( P(W_n^\Psi(t) \in D(A)) = 1 \).

For any \( n \in \mathbb{N} \), \( t \geq 0 \), we can estimate

\[
|A_n W_n^\Psi(t) - AW^\Psi(t)|^2_H \leq 3|N_{n,1}(t) + N_{n,2}(t)|,
\]

where

\[
N_{n,1}(t) := |A_n W_n^\Psi(t) - A_n W^\Psi(t)|_H, \\
N_{n,2}(t) := |A_n W^\Psi(t) - AW^\Psi(t)|_H.
\]

Using the convergence of resolvents (5) and the Yoshida approximation properties, we can follow the same steps as above for proving

\[
\lim_{n \to +\infty} \sup_{t \in [0, T]} \mathbb{E}(N_{n,1}(t)) \to 0
\]

and

\[
\lim_{n \to +\infty} \sup_{t \in [0, T]} \mathbb{E}(N_{n,2}(t)) \to 0.
\]

Therefore, we can deduce that

\[
\lim_{n \to +\infty} \sup_{t \in [0, T]} \mathbb{E}|A_n W_n^\Psi(t) - AW^\Psi(t)|^2_H = 0,
\]

and then (13) holds. \( \square \)
Theorem 2. Suppose that assumptions of Proposition 3 hold. Then the equation (1) has a strong solution. Precisely, the convolution \( W^\Psi \) given by (12) is the strong solution to (1).

Proof. In order to prove Theorem 2 we have to show only the condition (10). Let us note that the convolution \( W^\Psi \) has integrable trajectories. Because the closed unbounded linear operator \( A \) becomes bounded on \( (D(A), \| \cdot \|_{D(A)}) \), see [4] Chapter 5, we obtain that \( AW^\Psi (\cdot ) \in L^1 ([0, T]; H) \), P-a.s. Hence, properties of convolution provide integrability of the function \( a(T-\tau)AW^\Psi (\tau) \) with respect to \( \tau \), what finishes the proof.

3. Fractional Volterra equations. As we have already written, [2] contains some class of equations. For instance when \( a(t) = \frac{t^{\alpha-1}}{1^{(\alpha)}} \), \( \alpha > 0 \), we obtain integro-differential equations studied by many authors, see e.g. [1] and references therein. These facts lead us to the fractional stochastic Volterra equations of the form

\[
X(t) = X(0) + \int_0^t g_\alpha(t-\tau)AX(\tau)d\tau + \int_0^t \Psi(\tau)dW(\tau), \quad t \geq 0,
\]

where \( g_\alpha(t) = \frac{t^{\alpha-1}}{1^{(\alpha)}} \), \( \alpha > 0 \). Let us emphasize that for \( \alpha \in (0, 1) \), \( g_\alpha \) are completely positive, but for \( \alpha > 1 \), \( g_\alpha \) are not completely positive.

Now, the pairs \((A,g_\alpha(t))\) generate \( \alpha \)-times resolvents \( S_\alpha(t), \ t \geq 0 \) which are analogous to resolvents defined in section 1 for more details, see [1].

Remark 2. Observe that the \( \alpha \)-times resolvent family corresponds to a \( C_0 \)-semigroup in case \( \alpha = 1 \) and a cosine family in case \( \alpha = 2 \). (Let us recall, e.g. from [3], that a family \( C(t), t \geq 0 \), of linear bounded operators on \( H \) is called cosine family if for every \( t, s \geq 0 \), \( t > s \): \( C(t+s) + C(t-s) = 2C(t)C(s). \)) In consequence, when \( 1 < \alpha < 2 \) such resolvent families interpolate \( C_0 \)-semigroups and cosine functions. In particular, for \( A = \Delta \), the integro-differential equations corresponding to such resolvent families interpolate the heat equation and the wave equation, see, e.g. [6].

We consider two cases:

(A1): \( A \) is the generator of \( C_0 \)-semigroup and \( \alpha \in (0, 1) \);

(A2): \( A \) is the generator of a strongly continuous cosine family and \( \alpha \in (0, 2) \).

In this part of the paper, the results concerning a weak convergence of \( \alpha \)-times resolvents play the key role. Using the very recent result due to Li and Zheng [10], we can formulate the approximation theorems for fractional Volterra equations.

Theorem 3. Let \( A \) be the generator of a \( C_0 \)-semigroup \((T(t))_{t \geq 0}\) in \( H \) such that \( \| T(t) \| \leq M e^{-t}, \ t \geq 0 \). Then, for each \( 0 < \alpha < 1 \) there exist bounded operators \( A_n \) and \( \alpha \)-times resolvent families \( S_{\alpha,n}(t) \) for \( A_n \) satisfying \( \| S_{\alpha,n}(t) \| \leq M C e^{(2\alpha)^{1/\alpha} t} \), for all \( t \geq 0 \), \( n \in \mathbb{N} \), and

\[
S_{\alpha,n}(t)x \to S_{\alpha}(t)x \quad \text{as} \quad n \to +\infty
\]

for all \( x \in H, t \geq 0 \). Moreover, the convergence is uniform in \( t \) on every compact subset of \( \mathbb{R}_+ \).

Outline of the proof. The first assertion follows from [1] Theorem 3.1], that is, for each \( 0 < \alpha < 1 \) there is an \( \alpha \)-times resolvent family \((S_{\alpha}(t))_{t \geq 0}\) for \( A \) given by

\[
S_{\alpha}(t)x = \int_0^\infty \varphi_{t,\alpha}(s)T(s)x ds, \quad t > 0,
\]

where \( \varphi_{t,\alpha} \) is the \( \alpha \)-times resolvent kernel.
where \( \varphi_{t,\gamma}(s) := t^{-\gamma}\Phi_{\gamma}(st^{-\gamma}) \) and \( \Phi_{\gamma}(z) \) is the Wright function defined as
\[
\Phi_{\gamma}(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1.
\]
Define
\[
A_n := nAR(n, A) = n^2R(n, A) - nI, \quad n > w,
\]
the Yosida approximation of \( A \).
Since each \( A_n \) is bounded, it follows that for each \( 0 < \alpha < 1 \) there exists an \( \alpha \)-times resolvent family \( (S_{\alpha,n}(t))_{t \geq 0} \) for \( A_n \) given as
\[
S_{\alpha,n}(t) = \int_0^\infty \varphi_{t,\alpha}(s)e^{sA_n}ds, \quad t > 0.
\]
We recall that the Laplace transform of the Wright function corresponds to \( E_{\gamma}(z) \) where \( E_{\gamma} \) denotes the Mittag-Leffler function. In particular, \( \Phi_{\gamma}(z) \) is a probability density function. It follows that for \( t \geq 0 \):
\[
\|S_{\alpha,n}(t)\| \leq \int_0^\infty \varphi_{t,\alpha}(s)||e^{sA_n}||ds \\
\leq M\int_0^\infty \varphi_{t,\alpha}(s)e^{2\alpha s}ds = M\int_0^\infty \Phi_{\alpha}(\tau)e^{2\omega\alpha \tau}d\tau = ME_\alpha(2\omega t^{\alpha}).
\]

The continuity in \( t \geq 0 \) of the Mittag-Leffler function and its asymptotic behavior, imply that for \( \omega \geq 0 \) there exists a constant \( C > 0 \) such that
\[
E_\alpha(\omega t^{\alpha}) \leq Ce^{\omega^{1/\alpha}t}, \quad t \geq 0, \alpha \in (0, 2).
\]

This gives
\[
\|S_{\alpha,n}(t)\| \leq MCe^{(2\omega)^{1/\alpha}t}, \quad t \geq 0.
\]

Now we recall the fact that \( R(\lambda, A_n)x \to R(\lambda, A)x \) as \( n \to \infty \) for all \( \lambda \) sufficiently large (e.g. [11] Lemma 7.3), so we can conclude from [10] Theorem 4.2] that
\[
S_{\alpha,n}(t)x \to S_\alpha(t)x \quad as \quad n \to +\infty
\]
for all \( x \in H \), uniformly for \( t \) on every compact subset of \( \mathbb{R}_+ \).

An analogous convergence for \( \alpha \)-times resolvents can be proved in another case, too.

**Theorem 4.** Let \( A \) be the generator of a \( C_0 \)-cosine family \( (T(t))_{t \geq 0} \) in \( H \). Then, for each \( 0 < \alpha < 2 \) there exist bounded operators \( A_n \) and \( \alpha \)-times resolvent families \( S_{\alpha,n}(t) \) for \( A_n \) satisfying \( \|S_{\alpha,n}(t)\| \leq MCe^{(2\omega)^{1/\alpha}t} \), for all \( t \geq 0 \), \( n \in \mathbb{N} \), and \( S_{\alpha,n}(t)x \to S_\alpha(t)x \) as \( n \to +\infty \) for all \( x \in H \), \( t \geq 0 \). Moreover, the convergence is uniform in \( t \) on every compact subset of \( \mathbb{R}_+ \).

Now, we are able to formulate the result analogous to that in section [2]

**Theorem 5.** Assume that (A1) or (A2) holds. If \( \Psi \) and \( A\Psi \) belong to \( N^2(0, T; L^0) \) and in addition \( \Psi(t)(U_0) \subset D(A) \), P-a.s., then the equation [7] has a strong solution. Precisely, the convolution
\[
W^\Psi_\alpha(t) := \int_0^t S_\alpha(t - \tau)\Psi(\tau)dW(\tau)
\]
is the strong solution to [7].
Outline of the proof. First, analogously like in section 2 we show that the convolution $W_\alpha^\Psi(t)$ fulfills the following equation

$$W_\alpha^\Psi(t) = \int_0^t g_\alpha(t-\tau) A W_\alpha^\Psi(\tau) d\tau + \int_0^t \Psi(\tau) dW(\tau).$$  \hspace{1cm} (24)

Next, we have to show the condition

$$\int_0^T |g_\alpha(T-\tau) A W_\alpha^\Psi(\tau)|_H d\tau < +\infty, \ P - a.s., \ \alpha > 0,$$  \hspace{1cm} (25)

that is, the condition (10) adapted for the fractional Volterra equation (22).

The convolution $W_\alpha^\Psi(t)$ has integrable trajectories, that is, $W_\alpha^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. The closed linear unbounded operator $A$ becomes bounded on $(D(A), \cdot|_{D(A)})$, see [14, Chapter 5]. Hence, $A W_\alpha^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. Therefore, the function $g_\alpha(T-\tau) A W_\alpha^\Psi(\tau)$ is integrable with respect to $\tau$, what completes the proof.

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