Discrete mathematics and mathematical foundations of computer science
Exercise Sheet No. 1

(1) An SSN is a 9-digit number with zeroes allowed in every position.
  (a) How many SSNs are there all of whose digits are even?
  (b) How many SSNs are a multiple of 2?
  (c) How many SSNs are there all of whose digits are distinct?
  (d) How many SSNs are there whose digits are in strictly increasing order.
(2) How many n-digit numbers are there, $n > 1$?
(3) How many n-digit numbers ($n > 1$)in which no neighbor digits are the same are there?
(4) How many numbers in $[1, 10^n]$ whose no neighbor digits are the same are there ($n > 1$)?
(5) How many n-digit numbers represent the same number regardless of the reading direction ($n > 1$)?
(6) A contest has 5 prizes: TV, iPod, cellphone, bicycle, vacation. You get to pick 2 of them, except that you cannot pick two electronic goods. How many possibilities do you have?
(7) A chromosome contains many genes. The genes do not overlap, and each gene can be oriented either forward or back.
  (a) How many chromosomes in which 10 genes must appear in order: from 1 up to 10 are there?
  (b) How many chromosomes with 10 genes are there?
(8) In how many ways can you choose 6 cards from the set of 52 cards, so that there are exactly 2 peaks among them?
(9) In how many ways can you choose 6 cards from the set of 52 cards, so that there are at least 2 peaks among them?
(10) In how many ways can you choose 6 cards from the set of 52 cards, so that there are cards of all four colors among them?
(11) In how many ways can you choose four numbers from [301], so that their sum is even?
(12) How many 9-digit numbers can be composed from digits $2, 2, 2, 3, 3, 3, 4, 4, 4, 4$?
(13) In how many ways can 46 tickets be distributed between 19 people if the tickets are
  (a) for the same concert?
  (b) for various concerts?
(14) In how many ways can you solve the equation
$x_1 + \ldots + x_n = k$ if $n, k \in \mathbb{N}$ and
  - $\forall_{i\in[n]} x_i \in \mathbb{N}_0$,
  - $\forall_{i\in[n]} x_i \in \mathbb{N}$
(15) How many n-digit positive numbers that have the sum of digits less than 9 are there?
(16) How many n-digit positive numbers that have digits in non-decreasing order are there?
(17) We select randomly a number from the set [10000]. Find the probability that this is a number:
  - divisible by 6 or divisible by 15,
  - indivisible by 6 or indivisible by 15,
  - divisible by at least one of: $6, 21, 5$,
  - indivisible by at least one of: $6, 21, 5$.

(18) Let A be an arbitrary 10-element subset of $[50]$. Show that A contains two different 5-element subsets such that both of them have the same sum of elements.

(19) n teams participate in a football tournament. Every two of them have to play one match. In the following days matches are played according to a timetable. Show that at the end of every tournament's day there are two teams that have played the same number of matches so far.

(20) There are 30 students in the class. Adam made 13 mistakes in the dictation and others made less mistakes than him. Prove that there are at least 3 students in the class who did the same number of mistakes in the dictation.

(21) Imagine a matrix of size $n \times n$ consisting of elements: $-1, 0, 1$. Is it possible that the sum of elements in all lines (rows, columns and both diagonals) are different?

(22) Let A be a 25-element subset of the set $[150]$. Show that there are two disjoint 2-element subsets of A that have the same sum of the elements.

(23) Suppose that every of 925 people points out exactly one solution of the equation $x_1 + \ldots + x_5 = 12$ such that the solution satisfies the conditions: $x_2 \geq 3$ i $x_5 \geq 2$ and $x_1, \ldots x_5 \in \mathbb{N}_0$. Show that there are at least 3 people who have pointed out the same solution.

(24) In how many orders can you put vertically 3 identical white flags, 4 identical yellow flags and 4 identical blue flags?

(25) How many anagrams can you make from the letters $a, a, b, b, b, c, d, e, f, g, h, k, k, k, k, m$ (anagrams have not be meaningful or even pronouncible)?

(26) How many anagrams can you make from the letters $a, a, b, b, b, c, d, e, f, g, h, k, k, k, k, m$ in such a way that they contain no sequence h, e, g (anagrams have not be meaningful or even pronouncible)?

(27) How many anagrams can you make from the letters $a, a, b, b, b, c, d, e, f, g, h, k, k, k, k, m$ in such a way that they contain no sequence h, e, g and no sequence f, a, c (anagrams have not be meaningful or even pronouncible)?

(28) How many anagrams can you make from the letters $a, a, b, b, b, c, d, e, f, g, h, k, k, k, k, m$ in such a way that they contain no sequence h, e, g and no sequence f, a, c and no sequence m, a, d (anagrams have not be meaningful or even pronouncible)?

(29) Each of three families have five children. All 15 children camp in five triple tents by the lake. Assuming that the placement of children is accidental, count the probability that each family has at least 2 children in one of the tents.

(30) We create words (not necessarily meaningful) from 12 letters each of which is in $\{a, b, c, d\}$ (repetitions are possible). In how many ways can we do it?

(31) We create sets consisting of 12 letters each of which is in $\{a, b, c, d\}$ (repetitions are possible). In how many ways can we do it?

(32) The number of different components obtained after writing the expression $(a + b + c + d)^2$ in details is equal to 10. Namely, the components are: a^2, b^2, c^2, d^2, $2ab$, $2ac$, $2ad$, $2bc$, $2bd$, $2cd$. Write the number of components of the expression $(a + b + c + d)^n$ as a function of variable n.

(33) Prove the following identities (if it is possible use the combinatorial interpretation of both sides):
   (a) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
   (b) $\binom{n}{k} = \binom{n}{n-k}, \ k \leq n$
   (c) $\sum_{k=0}^{n} \binom{n}{k}(-1)^k = 0$
   (d) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \ n, k \in \mathbb{N}$

(34) Let X be a non-empty finite set. Show that the family of subsets of X whose each members has even number of elements has the same cardinality as the family of subsets of X whose each members has odd number of elements (0 is even).

(35) Let $a_n = 3a_{n-1} + 4a_{n-2}$, $n \geq 2$ and $a_0 = 0$, $a_1 = 5$. Find the description of a_n as a function of n.

(36) Let a_n be the number of ternary strings of length n, in which there is no repetition of the symbol 1. Find the recurrence formula on a_n and next the formula being a function of n.

(37) Let a_n be the number of ternary strings of length n, in which there is no repetition of the symbol 1 and there is no repetition of the symbol 2. Find the recurrence formula on a_n and next the formula being a function of n.

(38) Let a_n be the number of ternary strings of length n, in which there is no repetition of the symbol 1 or there is no repetition of the symbol 2. Find the recurrence formula on a_n and next the formula being a function of n.

(39) How many strings (A_1, A_2, A_3) such that $A_1 \cup A_2 \cup A_3 = [n]$ are there?

(40) In how many ways can you put vertically on n-meter mast flags in three different colours if red flags are 2-meter wide and remaining flags are 1-meter wide (we have as many flags as we need in each colour)?

(41) Inside the nuclear reactor, there are two types of components: α and β. In each second the β component breaks down into three α components and the α components breaks down into one α comonent and two β components. If we place in the reactor one β component at the initial moment ($t = 0$), then how many α components will be at the t-moment for $t = 2018$?

(42) Let $p \in \mathbb{N}$, $p \geq 3$, be a constant. Describe which objects can we count using the sequence $(a_n)_{n=0}^{\infty}$ if a generating function $f(x)$ for the sequence has a form:
- $\frac{1}{(1-x)^p} = (1 + x + x^2 + \ldots)^p$,
- $\frac{x^p}{(1-x)^p} = (x + x^2 + \ldots)^p$,
- $(x^2 + x^4 + x^6 + \ldots)(x + x^2 + x^3 + x^4 \ldots)^{p-1}$.

(43) How many different ways can you get the sum of 31 points on 26 different dices? How many different ways can you get the sum of n points on 26 different dices?

(44) Find a generating function for a sequence $(a_n)_{n=1}^{\infty}$ that describes the number of solutions of the equation $21x + 5y + 17z = n$ if the equation satisfies the conditions $x, y, z \in \mathbb{N}$, $2 \leq x \leq 11$, $1 \leq y \leq 7$, $z > 4$.

(45) Use a generating function in order to find the number of n-element sets composed from B, C, D, E, F (repetitions are acceptable), in which letters B and E can be repeated at most twice and remaining letters can be repeated as many times as we wont.

(46) In how many ways can you build an n-high tower with blocks if you have an unlimited number of blocks in three colors: red, yellow, blue?

(47) In how many ways can you build an n-high tower with blocks if you have an unlimited number of blocks in three colors: red, yellow, blue and moreover, red and blue blocks there are not neighbours in the tower?

References:

- K.A. Ross, Ch.R.B. Wright, *Discrete mathematics* ,
- Praca zbiorowa pod redakcją K.A. Rybnikowa, *Analiza kombinatoryczna w zadaniach.*