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On non-negative recursive utilities in dynamic programming with nonlinear aggregator and CES.

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Abstract It is studied uniqueness and global attracting property of the recursive utility under uncertainty related to Epstein and Zin [19] equations. The equation is specified by a temporal aggregator W which satisfies different conditions then Marinacci and Montrucchio [40], Le Van and Vailakis [36] and Jaśkiewicz, Matkowski and Nowak [30]. The random continuation value is parametrized by non-linear Certainty Equivalent Substitution (CES-for short). There are studied a properties of truncation error, operating on a solid normal cone of function space. The main results are applied to the theory of optimal economic growth model (related with resource extraction game) with nonlinear aggregator for instance, but they can be applicable in other economic models.
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1 Introduction

In many economic problems, the Böhm-Bawerks idea of preference for advancing the timing of future satisfaction from different point of view makes important role. This idea has been formalized by Koopmans [33], where it has been characterized axiomatically a class of recursive utilities. For the extension of Koopmans result to the models under uncertainty, the reader is referred to Kreps and Porteus [35] for finite horizon model and Epstein and Zin [19] for infinite horizon model. This class includes a classical discounting utility of Samuelson [48] and some models of intergenerational altruism, postulated by Ramsey [45].

Many researchers have used standard discounting rule to various sequential decision-making problems, see Blackwell [10]; Strauch [49]; Lucas et. all [38]; Hernandez-Lerma and Lasserre [27]; and references therein. In all aforementioned works, in order to obtain a solution of the Bellman equation, one can directly apply the Banach Contraction Principle, if the immediate return function is bounded. In case of unbounded instantaneous utility functions, one may combine with a weighted norm approach, see Becker and Boyd [7]; Boyd [11]; Hernandez-Lerma and Lasserre [28], with an idea of local contractions of Rincón-Zapatero and Rodriguez-Palmero [46][1]. In some papers like Marinacci and Montrucchio [40], Montrucchio [43], Martins-da-Rocha and Vailakis [22] a Thompson metric (see [53]) has been used on a set of comparable functions as an alternative for standard sup-norm. In turn in Jaśkiewicz et. all [30] unbounded utility has been obtained as a limit of corresponding recursive utilities in the models with bounded returns.

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[1] See also Rincón-Zapatero and Rodriguez-Palmero [47], Nowak and Matkowski [42], Martins-da-Rocha and Vailakis [22].

Banach Contraction Principle is applicable with more general recursive utilities like in Denardo [16]; Lucas and Stokey [37], Boyd [11] and with Blackwell and Thompson aggregator in Marinacci and Montrucchio [40]. In turn in Jaśkiewicz et. all [30] a Matkowski extension of Banach Principle [41] has been used.

In this paper it is considered multi-stage model in which single agent has a preference depending on the utilities derived at any stage. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of random utilities derived by the agent in a sequential decision-making process and W is a fixed aggregator. Then, it is studied expected total utility, which is on the form

$$J := \lim_{n\to\infty} \left(W \left(u_1, \mathcal{M} \left(W \left(u_2, \ldots, \mathcal{M} \left(W \left(u_{n-1}, \mathcal{M} \left(W(u_n, 0) \right) \right) \right) \right) \right) \right) \right), \tag{1}$$

where W is an aggregator, and $\mathcal{M}(\cdot)$ is *certainty equivalent substitution* (CES for short). CES represents uncertainty between periods or generations and it usually depends on the transition probability. This total utility is in a spirit of Koopmans [33] and Epstein and Zin [19]. In standard dynamic programming an aggregator, W has affine form $W(u, v) = u + \beta v$ and $\mathcal{M}$ is standard expectation. But the preferences represented by expectation argue with Allais paradox[2]. Because of that $\mathcal{M}$ can be express more generally then expectation i.e. from a Chew - Dekel class [13,15] at the form $\mathcal{M}(\cdot) = \phi^{-1}(E\phi(\cdot))$ with some strictly monotone and continuous convex or concave function ϕ. In particularly if $\phi(x) = e^{-x}$ then CES is said to be *risk sensitive* (see [54]). Risk sensitivity has been postulated in Weil [54] and used in Jaśkiewicz and Nowak [29] in OLG models and in Bäuerle and Jaśkiewicz [6]. To the other CES, representing different preferences and remeding Allais paradox the reader is referred to Gul [21], Dekel [15], Chew and Epstain [14] for example.

For other then affine aggregator the reader is referred to Jaśkiewicz et. all [30], where a constant discount factor β has been replaced by discounting function δ i.e. aggregator has a form $W(u, v) = u+\delta(v)$. The logarithmic aggregator has been considered by Koopmans, Diamond, and Williamson [34], and large class of Blackwell and Thompson aggregators has been considered by Marinacci and Montrucchio [40] or Martins da Rocha and Vailakis [22]. Under mild conditions, and nonnegative immediate utility function it is possible to use standard Tarski-Kantorovich Theorem (see Dugundji and Granas [17] and Tarski [52]) to prove that the limit in (1) exists and it is recursive utility. The issue is to prove that utility in (1) is unique and obeys attracting property. In standard discounted models and its extensions to other aggregators like in Denardo [16]; Lucas and Stokey [37]; Boyd [11]; Marinacci and Montrucchio [40]; Jaśkiewicz et. all [30]; this problem has been solved under specific assumptions. Observe that, in case of possibly negative utilities the limit in (1) may not exist and must be replaced by $\liminf$ or $\limsup$ (for example see Bich et all [9]).

In this paper it is studied existence and global attracting property of recursive utility in a form (1), under nonlinear aggregator, sub-homogenous CES and distinct conditions then Jaśkiewicz et. all [30] and Marinacci and Montrucchio [40]. It is proven the existence, uniqueness and attracting property of a solution of Bellman equation in an infinite time horizon models with a fixed real-valued aggregator W. The analysis are applicable with nonnegative and either bounded and unbounded from above immediate return functions. In unbounded case, it is neither applied weighted norm nor local contraction idea from [46]. Instead, it is successively extended the state space between invariant subsets of state space corresponding this law of motion. The point is, the construction of law of motion in this paper prevents to extend some level, since for sufficiently large state the production pull the capital down (similar law of motion is in [50], [51] or [39]). This question for unbounded utilities from bellow is however important as a similar models, but with a constant discount factor, were considered by Strauch [49]; Hernandez-Lerma and Lassere [27] or seminal paper of Lucas et. all [38]. For a further discussion see also Feinberg [20], and more general models with unbounded return [36]; [40]; [30] and in [9].

[2] Let us denote $(p_1, x_1; \ldots; p_k, x_k)$ as a lottery with output x_i with probability p_i $(i = 1, \ldots, k)$. Allais paradox states that lottery $l_1 := (1, 100)$ is more preferable then $l_2 := (0.8, 200; 0.2, 0)$ for most people, while lotery $l_3 = (0.5, 100; 0.5, 0)$ is less preferable then $l_4 = (0.4, 200; 0.6, 0)$. Observe that $l_3 = \frac{1}{2}l_1 + \frac{1}{2}\delta_0$, $l_4 = \frac{1}{2}l_2 + \frac{1}{2}\delta_0$, where $\delta_0 = (1, 0)$ Hence utility on lotteries representing this preference can not be expectation

In this paper it is applied alternative technique in the proofs, applying Guo, Cho and Zhu [25] fixed point Theorems, on a cone of nonnegative functions. Guo, Cho and Zhu fixed point theorem enables us to have uniqueness and global attracting property of the recursive utility function, which mathematically is a fixed point of some increasing operator. To motivate this result it is shown examples where neither Banach Contraction Principle and nor its extension due to Matkowski [41] can be used. Similarly as Banach Contraction Principle, Guo Cho and Zhu Theorem delivers us a truncation error convergent to zero as fast as exponential function. This theorem has already been used in OLG models e.g. Balbus et. all [5] and [4]. Existence, uniqueness and global attracting property of recursive utility function enables us to obtain an existence of optimal plan as well as properties of optimal recursive utility function. Unfortunately, the main results exclude many useful aggregators as affine, logarithmic [34] and large class of Thompson aggregators. On the other hand all aforementioned aggregators can be described us limit of these considered in this paper. Because of that, we comment the possibilities of approaching those models by the models considered in this paper.

The rest of the paper is structured as follows: Section 2 contains preliminaries on fixed point theorem on solid normal cones, which may be alternative tool for standard Banach Contraction Principle and for its extension due to Matkowski [41]. Section 3 contains description of the model with main assumptions. Main results are contained in Sections 4 and 5. In Section 4 existence, uniqueness and global attracting property of recursive utility is proven, and in Section 5 it is proven existence of optimal policy using Bellman equations. In Section 6 we describe the problems with the models satisfying more general assumptions then in this paper. For example in Subsection 6.2 the problem with no optimal policy is presented. In turn in Subsection 6.3 we comment how to approach many models by the model satisfying our assumptions. The last section contains concluding remarks.

2 Preliminaries

2.1 Fixed point theorems on solid normal cone

Let $(V, ||\cdot||)$ be a Banach space with $\mathbf{0} \in V$ as its *zero vector*.

Definition 1 A subset $P \subset V$ is said to be **cone** if following axioms are satisfied:

- if $v \in P$, $t \in \mathbb{R}_+$ then $tv \in P$,
- if $v \in P$, $-v \in P$ then $v = \mathbf{0}$.

Each cone generates partial order relation $\leq_P$ in the following way: $v \leq_P w$ iff $w - v \in P$. In this paper let us drop P from $\leq_P$ and let us use common notation $\leq$ to indicate an order generating on the cone as well as to indicate standard order on the real line.

Definition 2 A cone P is said to be **solid** if its interior P^o is nonempty.

Definition 3 A cone P is said to be **normal** if there is some positive number N such that for all $v \in P$ and $w \in P$

$$\text{if } \mathbf{0} \leq v \leq w \text{ then } ||v|| \leq N||w||.$$

N is said to be **index of normality**.

Definition 4 Let $(V, ||\cdot||)$ be a Banach space. Let $V_0 \subset V$ and $T : V_0 \to V_0$ be some operator. Let $v^* \in V_0$ be a fixed point of T. v^* is said to have **global attracting property** on V_0 if for all $v_0 \in V_0$ $\lim_{n\to\infty} ||T^n v_0 - v^*|| = 0$ ($T^n := T \circ \ldots \circ T$ means n-th composition of T).

In entire paper *increasing* means order preserving function i.e. $x \leq y$ implies $f(x) \leq f(y)$.

Theorem 1 (Theorem 3.1.7. in [25]) *Let $(V, ||\cdot||)$ be a Banach space. Assume $P \subset V$ is solid and normal cone generating order $\leq$ with N as an index of normality. Let $T : P^o \to P^o$ be increasing operator such that there exists $r \in (0,1)$ such that for all $v \in P^o$, $t \in (0,1]$ it holds*

$$T(tv) \geq t^r T(v).$$

Then T has unique fixed point $v^* \in P^o$, with global attracting property and estimation rate

$$||T^n v_0 - v^*|| \leq M\left(1 - \alpha^{r^n}\right) \text{ for all } n \in \mathbb{N}.$$

Here $M = 2N||v_0||$, $\alpha = \frac{t_0}{s_0}$ and t_0 and s_0 are chosen in a following way:

$$0 < t_0 < 1 < s_0 \text{ and it holds } t_0^{1-r} v_0 \leq T(v_0) \leq s_0^r v_0.$$

This theorem above yields distinct assumptions then standard Banach Contraction Theorem and its generalizations (see [41] or [17]). Theorem 1 gives the conditions for uniqueness and global attracting properties of fixed points. It is worth mentioning that operator T in Theorem 1 maps open set into itself. Because of that it can be find unique fixed point in the interior of its domain. Observe that extension of T in Theorem 1 to closure of P may have more fixed points. For example $T : \mathbb{R}_+ \to \mathbb{R}_+$ defined as $T(x) = \sqrt{x}$ has two fixed points, but only one in the interior. Hence neither Banach Contraction Theorem nor its extensions (like Matkowski [41]) are applicable in this case.

2.2 Basic notations and terminology in a space of functions and measures.

In this section we induce general notations which is used in entire paper.

- If Ω is a metric space
  - then $\mathcal{B}(\Omega)$ is a collection of Borel subsets of Ω,
  - $B(\Omega)$ is a set of all Borel measurable, bounded and real valued functions on Ω,
  - and it is Banach space with a sup-norm $||\cdot||_\Omega$ i.e.

$$||v||_\Omega := \sup_{\omega \in \Omega} |v(\omega)|, \tag{2}$$

  - by $[\cdot]_\Omega$ is an infimum operator on $B(\Omega)$ i.e.

$$[v]_\Omega = \inf_{\omega \in \Omega} v(\omega)$$

  for all $v \in B(\Omega)$.
- If Ω is Polish space then $\Delta(\Omega)$ is denoted as a set of all Borel probability measures on Ω. Endow $\Delta(\Omega)$ with a standard *weak topology* i.e. $\mu_n \to \mu$ as $n \to \infty$ if for all real valued continuous function f on Ω it holds

$$\lim_{n\to\infty} \int_\Omega f(\omega)\mu_n(d\omega) = \int_\Omega f(\omega)\mu(d\omega).$$

  Clearly $\Delta(\Omega)$ is Polish space (for example see Theorem 15.12 in [1]).

3 The model

3.1 Description of the model

We shall consider a dynamical system specified by the following objects $(X, \Gamma, \Omega, u, q, W, \mathcal{M})$ where:

- $X \subset \mathbb{R}$ denotes the space of all possible capital levels; assume $X = [0, \bar{x}]$ where $\bar{x} \in \mathbb{R}_+$ or $X = [0, \infty)$;
- For each $x \in X$ the correspondence $\Gamma(x) := [0, x]$ denotes the set of feasible investment levels for an agent when the current capital level is x;
- $\Omega := [\underline{\omega}, \overline{\omega}]$ (here $0 < \underline{\omega} \leq \overline{\omega}$) is a space of random shocks endowed with a Borel probability measure ρ;
- The function $q : X \times \Omega \to \mathbb{R}$ denotes a random production function;
- $u : X \to \mathbb{R}_+$ is a *instantenous utility function* (one-period utility);
- $W : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is called *aggregator function*; assume it is increasing in both arguments;
- $\mathcal{M}^q : B(X) \to B(X)$ is *certainty equivalent substitution*[3] (CES) depending on q. In entire paper we will denote $\mathcal{M}$ instead of $\mathcal{M}^q$ for short;

In another words it is considered a model in which a decision maker chooses a consumption level in periods $n \in \mathbb{N}$. If $x_1 \in X$ is an initial capital level then this agent decides how much to invest and how much to consume. The investment level y_1 is chosen from the set $\Gamma(x_1) := [0, x_1]$ and the remaining part of this capital $c_1 := x_1 - y_1$ is a *consumption level*. Then, the *immediate return* for this agent $u(c_1)$ is generated, and the next capital level is produced by the function $x_2 = q(y_1, \omega_1)$, where $\omega_1 \sim \rho$ [4] is a shock unobservable at state 1. In state x_2 again two things happen: the agent selects an investment level $y_2 \in [0, x_2]$ and consumption level $c_2 := x_2 - y_2$ and the return $u(c_2)$ is incurred. Next capital level is updating by the random production function $x_3 = q(y_2, \omega_2)$ where ω_2 is unobservable shock at step 2, independent and having distribution ρ which is the same as ω_1. This procedure repeats itself yielding the history on the capital - investment system $(x_n, y_n)_{n\in\mathbb{N}}$. At the same time a sequence of independent random variables having distribution ρ, $(\omega_n)_{n\in\mathbb{N}}$ is generated.

Let H be a set of all feasible histories. Mathematically H is a set of all sequences $h := (x_n, y_n)_{n=1}^{\infty} \in Gr(\Gamma)^{\infty}$. Endow H with a natural Borel product $\sigma-$ algebra on $(Gr(\Gamma))^{\infty}$. For $n > 1$, we denote $H_n := Gr(\Gamma)^{n-1}$ as the set of feasible histories before step n. A *policy* is a sequence of jointly Borel measurable [5] mappings such that $\sigma_1 : X \to \Delta(Y)$, $\sigma_1(\Gamma(x)|x) = 1$, and for $n > 1$ $\sigma_n : H_n \times X \to \Delta(Y)$ such that for each $(h_n, x) \in H_n \times X$ it holds $\sigma_n(\Gamma(x_n)|h_n, x_n) = 1$. Let Σ be denoted as a set of all policies. A policy σ is said to be *pure* if for each $n \in \mathbb{N}$, $h_n \in H_n$, $x \in X$ there is some $y \in Y$ such that $\sigma_n(\{y\}|h_n, x) = 1$. A *Markov policy* is such that $\sigma_n : X \to \Delta(Y)$. $\sigma \in \Sigma$ is a *stationary Markov policy* if $\sigma_n = s$ ($n \in \mathbb{N}$) for some Borel measurable function $s : X \to \Delta(Y)$. Stationary Markov policy is identified with s. Let $x \in X$ be an initial state and $(\sigma_n)_{n\in\mathbb{N}}$ be arbitrary policy. By Ionescu-Tulcea Theorem [44] a production function q, initial capital $x \in X$ and policy σ induces unique probability measure P_x^{σ} on H.

Let Σ be a set of all policies. For each $\sigma \in \Sigma$, and $n > 1$ let us denote $\sigma^n := H_n \to \Sigma$ as $\sigma^n := (\sigma_{n+\tau})_{\tau=0}^{\infty}$ called $n-$ th *shift policy* i.e. policy from the period n onward. Observe that σ is Markov policy if and only if for each $n \geq 1$ the σ^n does not depend on H_n (i.e. is "constant" strategy). Let us state first assumption on CES which is necessary for all further analysis.
In the entire paper we slightly abuse notation and we will denote $\mathcal{M}_y(\cdot) := \mathcal{M}(\cdot)(y)$.

Assumption 1 *Let $k \in \mathbb{N}$ and $Z \in \mathcal{B}(\mathbb{R}^k)$ and suppose that the function $f : X \times Z \to \mathbb{R}$ is jointly measurable. Then*

$$(y, z) \in X \times Z \to \mathcal{M}_y(f(\cdot, z))$$

is jointly measurable function.

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[3] $\mathcal{M}(\cdot)$ is certainty equivalent substitution if $\mathcal{M}(v_1) \leq \mathcal{M}(v_2)$ whenever $v_1 \leq v_2$ and $\mathcal{M}(\alpha) = \alpha$ for each constant α.

[4] real valued random variable having distribution ρ

[5] That is Borel measurable with respect to corresponding product topology.

Following Hansen and Sargent [26] for any initial state $x \in X$ and $\sigma \in \Sigma$ let us define a total utility for the agent as an approach of total utilities in n-stage models. More precisely the $n-$ stage total utility is defined recursively as follows:

$$J_1(x,\sigma) := \int_{\Gamma(x)} W(u(x-y),0)\sigma_1(dy|x),$$

and for each $n > 1$ let

$$J_n(x,\sigma) := \int_{\Gamma(x)} W\left(u(x-y), \mathcal{M}_y(J_{n-1}(\cdot, \sigma^2(x,y)))\right)\sigma_1(dy|x) \tag{3}$$

and the *total utility* (if exists) is defined as follows

$$J(x,\sigma) = \lim_{n\to\infty} J_n(x,\sigma). \tag{4}$$

Observe that, if $\sigma \in \Sigma$ then $(x',x,y) \in X \times H_2 \to J_1(x', \sigma^2(x,y))$ must be jointly measurable, hence by Assumption 1

$$(y',x,y) \in X \times H_2 \to \mathcal{M}_{y'}(J_1(\cdot, \sigma^2(x,y)))$$

must be jointly measurable. Consequently J_2 is jointly measurable. Hence, by induction we conclude the joint measurability of all J_n, hence also its supremum. Since W is increasing in second argument and $\mathcal{M}_y(\cdot)$ is increasing for each $y \in X$, hence we can show that $J_1 \le J_2 \le \dots J_n \le \dots$. As a result the limit in (4) always exists, although it may be infinite.
Observe that: if $W(v_1,v_2) = v_1 + \beta v_2$ and

$$\mathcal{M}_y(v) := \phi^{-1}\left(\int_{\Omega} \phi(v(q(y,\cdot)))d\rho(\cdot)\right)$$

with $\phi(\cdot) := id_X(\cdot)$ then J is standard $\beta-$ discounted utility function. If $\phi(x) = e^{-\theta x}$ for some $\theta > 0$, the CES is called *risk -sensitive.*

3.2 Basic assumptions and their literature review

In Sections 4 and 5 the following assumptions are satisfied:

Assumption 2 *Assume u is increasing , bounded, continuous function and* $u(0) \ge 0$.

Assumption 3 *Assume W is jointly continuous, $W(v_1,v_2) = 0$ if and only if $v_1 = v_2 = 0$ and additionally there exists $r \in (0,1)$ such that for all $v_1 \in [u(0),\infty)$, $v_2 > 0$ and $t \in (0,1)$ it holds*

$$W(v_1, tv_2) \ge t^r W(v_1,v_2); \tag{5}$$

Assumption 4 *On the transition function q assume for each $\omega \in \Omega$, $q(\cdot,\omega)$ is strictly increasing and continuous function such that $q(0,\omega) = 0$ for each $\omega \in \Omega$; Moreover, there exists non-decreasing function $K : \Omega \to \mathbb{R}_{++}$ such that $q(x,\omega) > x$ if $0 < x < K(\omega)$ and $q(x,\omega) \le x$ if $x \ge K(\omega)$.*

Assumption 5 *Assume for each $y \in X$, $\mathcal{M}_y$ satisfies:*

(i) $\mathcal{M}_y$ is sub-homogenous operator i.e.

$$v \in B(X), t \in [0,1] \text{ implies that } \mathcal{M}_y(tv) \ge t\mathcal{M}_y(v);$$

(ii) $\mathcal{M}_y$ *is* order continuous *operator: i.e. if* $(v_n)_{n\in\mathbb{N}}$ *is a monotone sequence of the elements of* $B(X)$ $v_n \to v$ *then*

$$\lim_{n\to\infty} \mathcal{M}_y(v_n) = \mathcal{M}_y(v);$$

(iii) *for each measurable functions* $v : X \to \mathbb{R}$ *and* $w : X \to \mathbb{R}$*, if* $v(\omega) = w(\omega)$ *for* $\rho q_y^{-1} - a.a.\ \omega \in \Omega$ [6] *then* $\mathcal{M}_y(w) = \mathcal{M}_y(v)$;

(iv) $\mathcal{M}_y(v)$ *is continuous in* $y \in X$ *whenever* v *is.*

A few comments are in order. Assumption 2 is standard. Assumption 4 means that production is generally disturbed by random noise. At each step the noise is independent on the noise at any other step. Classical example is Cobb-Douglas production function in case of deterministic model i.e. Ω is singleton. More general form can be found in [50] and [51]. Random production function satisfying Assumption 4 can be found many papers on the growth model like [12], [39], [18], [2] in multigenerational altruism [8], [3] and references therein. Classical CES is expectation operator. Expectation is homogenous and all other properties in Assumption 5 are satisfied. For example, consider more general *CES* proposed in Chew [13] or Dekel [15] at the following form:

$$\mathcal{M}_y(v) = \phi^{-1}\left(\frac{\int_X h(q(y,\omega))\phi(v(q(y,\omega)))\rho(d\omega)}{\int_X h(q(y,\omega))\rho(d\omega)}\right) =, \phi^{-1}\left(\frac{\int_X h(x')\phi(v(x'))\rho q_y^{-1}(dx')}{\int_X h(x')\rho q_y^{-1}(dx')}\right) \tag{6}$$

for some Borel-measurable strictly positive function h and strictly monotone function ϕ. All of this *CES* satisfies (ii)-(v) of Assumption 5, but subhomogenity is satisfied for some class of ϕ. Following lemma states applicability of this class.

Lemma 1 *Let* $h : X \to \mathbb{R}_{++}$ *be an integrable function,* $\eta > 0$ *and* $\phi : [0,\eta] \to \mathbb{R}_+$ *be continuous, strictly monotone function, and differentiable on* $(0,\eta)$ *with continuous and bounded derivative. Suppose that CES* $\mathcal{M}_y$ *has a form in (6). Assume additionally at least one of the following conditions holds*

(i) ϕ *is strictly increasing and the function* $\phi'(\phi^{-1}(\cdot))\phi^{-1}(\cdot)$ *is concave or*

(ii) ϕ *is strictly increasing and the function* $\phi'(\phi^{-1}(\cdot))\phi^{-1}(\cdot)$ *is convex.*

Then for each $y \in X$*, CES at the form* $\mathcal{M}_y(\cdot)$ *is subhomogenous and consequently satisfies Assumption 5.*

Proof We only show that $\mathcal{M}_y$ satisfies point (i) in Assumption 5. Let $v \in (B(X))^o$, $y \in X$ and define $f : (0,1) \to \mathbb{R}_+$ as follows $f(t) = \mathcal{M}_y(tv)$. Suppose that $\mathcal{M}_y(v)$ has a form (6) with h and ϕ given as in the assumptions of this lemma. By Lemma 5, $f(t) > 0$ for each $t \in (0,1)$. Note that, if f is differentiable then the subhomogenity is equivalent $\frac{f'(t)t}{f(t)} \le 1$ for all $t \in (0,1)$. Let ψ be a density w.r.t. measure ρq_y^{-1} in the following form

$$\psi(x) := \frac{h(x)}{\int_X h(x)\rho q_y^{-1}(dx)} \text{ with } x \in X.$$

Put E as expectation of ψ[7]. Observe that $f(t) := \phi^{-1}(E(\phi(tv)))$ and for $t > 0$

$$\frac{f'(t)t}{f(t)} = \frac{E(tv\phi'(tv))}{\phi'\left(\phi^{-1}(E\phi(tv))\right)\phi^{-1}(E(\phi(tv)))}.$$

Put $Z = \phi(tv)$. Then

$$\frac{f'(t)t}{f(t)} = \frac{E(\phi'(\phi^{-1}(Z))\phi^{-1}(Z))}{\phi'(\phi^{-1}(E(Z)))\phi^{-1}(E(Z))}. \tag{7}$$

[6] That is $\rho\left(\{\omega \in \Omega : v(q(y,\omega)) = w(q(y,\omega))\}\right) = 1$

[7] i.e. $E(w) := \int_X w(x)\psi(x)\rho q_y^{-1}(dx)$ for each $w \in B(X)$

Step 1. Suppose ϕ is strictly increasing and $\phi'\left(\phi^{-1}(\cdot)\right)\phi^{-1}(\cdot)$ is concave. By standard Jensen inequality we have

$$E(\phi'(\phi^{-1}(Z))\phi^{-1}(Z)) \leq \phi'(\phi^{-1}(E(Z))))\phi^{-1}(E(Z))). \tag{8}$$

Since ϕ is strictly increasing, hence ϕ' is strictly positive as well as ϕ^{-1}. Hence (8) together with (7) yields $\frac{f'(t)t}{f(t)} \leq 1$.

Step 2. Suppose ϕ is strictly decreasing and $\phi'\left(\phi^{-1}(\cdot)\right)\phi^{-1}(\cdot)$ is convex. If ϕ is strictly decreasing then also ϕ^{-1} and its derivative is strictly negative function. Since $\phi'\left(\phi^{-1}(\cdot)\right)\phi^{-1}(\cdot)$ is convex, hence and by standard Jensen inequality the equation (8) holds with opposite inequality. Since ϕ is strictly increasing, hence ϕ' is strictly negative, but ϕ^{-1} is strictly positive valued. Hence equation (8) with opposite inequality, together with (7) implies $\frac{f'(t)t}{f(t)} \leq 1$.

By Lemma 1 it is easy to verify that the class of CES at the form as in (6) with $\phi(t) = t^{\theta}$ with $\theta > 0$ satisfies Assumption 5. In the risk sensitive model $\phi(t) = e^{-\theta t}$ with $\theta > 0$ and $t > 0$. Then ϕ is strictly decreasing and $\phi^{-1}(\tau) = -\frac{1}{\theta}\ln(\theta\tau)$ for $\tau \in \left(0, \frac{1}{\theta}\right]$. Note that

$$\phi'\left(\phi^{-1}(t)\right)\phi^{-1}(t) = \theta t \ln(\theta t),$$

which is convex. Hence by Lemma 1 all conditions in Assumption 5 is satisfied. For more discussion on risk sensitive model the reader is referred to [29] and [6].

Finally, let us comment Assumption 3. Observe that standard affine aggregator does not obey Assumption 3, unless $u(0) > 0$. More details we will present in Subsection 6.3. On the other hand, the paper of Marinacci and Montrucchio [40] includes a few aggregators satisfying Assumption 3. For example a class of Thompson aggregators as:

$$W(v_1, v_2) = \left(v_1^{\xi} + \beta v_2^{\eta}\right)^{\frac{1}{p}},$$

where, $\xi > 0$, $0 < \eta < p$ and $\beta \in (0,1)$. Indeed, for each $v_1 > 0$, $v_2 > 0$ and $t \in (0,1)$ we have

$$\frac{\frac{\partial}{\partial v_2}W(v_1, v_2)}{W(v_1, v_2)}v_2 = \frac{\eta}{r}\frac{\beta v_2^{\eta}}{v_1^{\xi} + \beta v_2^{\eta}} \leq \frac{\eta}{r} < 1.$$

As a result this aggregator satisfies 3. Next aggregator is a modification of the aggregator in Koopmans et all [34]:

$$W(v_1, y_2) = \frac{1}{\theta}\log\left(1 + v_1^{\xi} + \beta v_2^{\eta}\right), \tag{9}$$

with $\theta > 0$, $\xi > 0$ and $\eta \in (0,1)$. We have

$$\frac{\frac{\partial}{\partial v_2}W(v_1, v_2)}{W(v_1, v_2)}v_2 = \eta\frac{\beta v_2^{\eta}}{(1 + v_1^{\xi} + \beta v_2^{\eta})\log(1 + v_1^{\xi} + \beta v_2^{\eta})} \leq \eta \sup_{x>0}\frac{x}{(1+x)\log(1+x)} < 1.$$

As a result Assumption 3 is satisfied also in this case. Observe however that, the aggregator in [34] satisfies $\eta = 1$.

Next example shows that Assumption 3 does not imply Thompson property (according to Marinacci and Montrucchio [40] terminology). Namely concavity at 0 is violated.

Example 1 Let $\psi : [0,\infty) \to [0,\infty)$ be defined as follows:

$$\psi(x) = \begin{cases} \sqrt[4]{x} & \text{if } x \in [0,1) \\ \sqrt{x} & \text{if } \quad x > 1 \end{cases} \tag{10}$$

and consider aggregator in the form $W(v_1, v_2) = \psi(v_1 + v_2)$. Clearly such W satisfies Assumption 3 since for each $t \in (0, 1)$

$$\psi(v_1 + tv_2) \geq \psi(t(v_1 + v_2)) \geq t^{\frac{1}{2}}\psi(v_1 + v_2).$$

Observe however that W is not concave in 0. Put $v_1 = 0.9$, $v_2 = 1.1$ and $t = 1/11$. Then $v_1 + v_2 = 2$ and $v_1 + tv_2 = 1$. Hence $W(v_1, tv_2) = 1$ but

$$tW(v_1, v_2) + (1 - t)W(v_1, 0) = \frac{1}{11}\sqrt{2} + \frac{10}{11}\sqrt[4]{0.9} \approx 1.04 > 1 = W(v_1, tv_2),$$

which contradics concavity in 0.

4 Existence and global attracting property of recursive utility function

For each $h \in H$ and $n \in \mathbb{N}$ let h_n be a natural projection of h on H_n. Define

$$\mathbf{U} := \{U : X \times \Sigma \to \mathbb{R} : U \text{ is bounded and for each } , \sigma \in \Sigma, n \in \mathbb{N}$$

$$\text{a function } U(x_n, \sigma^n(h_n)) \text{ is jointly measurable in } (h_n, x_n) \in H_n \times X\}.$$

Clearly $\mathbf{U}$ endowed with natural sup-norm topology

$$||U||_{X\times\Sigma} := \sup_{(x,\sigma)\in X\times\Sigma} |U(x, \sigma)|$$

is Banach space (more precisely it is closed subspace of Banach space of bounded functions on $X \times \Sigma$). Put $\mathbf{U}_+$ as a subset of nonnegative functions from $\mathbf{U}$. Clearly $\mathbf{U}_+$ is normal cone with unit index of normality. Moreover, it induces standard component-wise order. That is $U_1 \leq U_2$ iff $U_1(x, \sigma) \leq U_2(x, \sigma)$ for each $(x, \sigma) \in X \times \Sigma$.
By Lemma 5 in Appendix, it is shown that $\mathbf{U}_+$ is solid cone and

$$\mathbf{U}^o_+ := \left\{U \in \mathbf{U}_+ : \inf_{(x,\sigma)\in X\times\Sigma} U(x, \sigma) > 0\right\}. \tag{11}$$

According to [33], [7], [19] or [40] among others, we induce following definition:

Definition 5 $U^* \in \mathbf{U}_+$ is said to be *recursive utility function* if for any strategy $\sigma \in \Sigma$ and any initial state $x \in X$ it holds

$$U^*(x, \sigma) = \int_{\Gamma(x)} W\left(u(x - y), \mathcal{M}_y(U^*(\cdot, \sigma^2(x, y)))\right) \sigma_1(dy|x).$$

Observe that recursive utility function U^* (if exists) is a fixed point of the following operator:

$$T_W(U)(x, \sigma) := \int_{\Gamma(x)} W\left(u(x - y), \mathcal{M}_y(U(\cdot, \sigma^2(x, y)))\right) \sigma_1(dy|x).$$

For all $\delta \geq 0$ put

$$T^\delta_W(U)(x, \sigma) := \int_{\Gamma(x)} W\left(u_\delta(x - y), \mathcal{M}_y(U(\cdot, \sigma^2(x, y)))\right) \sigma_1(dy|x),$$

where $u_\delta(x - y) = \max\left(u(x - y), \delta\right)$. Obviously $u_0 \equiv u$. If $u(0) > 0$ then $T^\delta_W \equiv T_W$ for $\delta < u(0)$.

The purpose of this section is to construct utility function in (1) as *recursive utility function.* A following lemma is needed:

Lemma 2 *Assume 1, 2, 3, 4 and 5 and $u \in B(X)$. Then for each $\delta \geq 0$*

(i) T_W^δ *maps both* $\mathbf{U}_+$ *and* $\mathbf{U}_+^o$ *into itself;*
(ii) $T_W^\delta(\cdot)$ *is increasing operator, and for each* $U \in \mathbf{U}_+$, $T_W^\delta(U)$ *is decreasing in* δ*;*
(iii) If $\delta > 0$ *then* $T_W^\delta(\mathbf{0})(x,\sigma) \geq W(\delta,0) > 0$*;*
(iv) J *is well defined and if* U *is any fixed point of* T_W^δ *then* $J \leq U$*.*

Proof Let $U \in \mathbf{U}_+$ and $\delta \geq 0$.

Proof of (i). We show that $T_W^\delta(U) \in \mathbf{U}_+$. For each $n > 1$ let $h_n \in H_n$ and $x_n \in X$. First we show that $T_W^\delta(U)(x_n, \sigma^n(h_n))$ is jointly measurable in (h_n, x_n). Observe that, since $U \in \mathbf{U}_+$, hence $(h_{n+1}, x') \in H_{n+1} \times X \to U(x', \sigma^{n+1}(h_{n+1}))$ is jointly measurable, hence by Assumption 1 $\mathcal{M}_{y'}(U(\cdot, \sigma^{n+1}(h_{n+1}))$ is jointly measurable in $(h_{n+1}, y') \in H_{n+1} \times X$. Observe that

$$T_W^\delta(U)(x_n, \sigma^n(h_n)) = \int_{\Gamma(x_n)} W(u_\delta(x_n - y_n), \mathcal{M}_{y_n}(U(\cdot, \sigma^{n+1}(h_{n+1}))))\sigma_n(dy_n|h_n, x_n).$$

Hence $T_W^\delta(U)(x_n, \sigma^n(h_n))$ must be jointly measurable in $H_n \times X$. Hence T_W^δ maps $\mathbf{U}_+$ into itself. To show T_W^δ maps $\mathbf{U}_+^o$ into itself, observe that by (11) and Assumption 3 it holds $T_W(U)(x,\sigma) \geq W(0, [U]_{X\times\Sigma}) > 0$.

Proof of (ii) is easy by Assumptions 3 and 5.

Proof of (iii). By (ii) it holds $T_W^\delta(U)(x,\sigma) \geq W(\delta, [U]_{X\times\Sigma}) > 0$. Put $U = \mathbf{0}$, then $T_W^\delta(\mathbf{0})(x,\sigma) \geq W(\delta, 0) > 0$.

Proof of (iv). From part (i) it follows that all J_n are well defined. From part (ii) J_n increases in n. Indeed, $J_1 \geq \mathbf{0}$ and suppose $J_{k+1} \geq J_k$ for some integer k. Then $J_{k+2} = T_W(J_{k+1}) \geq T_W(J_k) = J_{k+1}$. Consequently, J_n increases in n, hence converges pointwise to J . We show that J is no greater then any fixed point of T_W. Let U be any fixed point of T_W. Observe that $U \geq \mathbf{0}$, and suppose $U \geq J_k$. Then by (ii) and induction hypothesis $U = T_W(U) \geq T_W(J_k) = J_{k+1}$, consequently U is greater then J.

Next lemma shows that, having capital level $x \in X$, we may ensure that our daily utility never pull down below some fixed value.

Lemma 3 *Assume 1, 2, 3, 4 and 5 and let* $x > 0$*. Then there exists* $\delta > 0$ *and a policy* $\sigma \in \Sigma$ *such that*

$$P_x^\sigma\left(\left\{h := (x_n, y_n)_{n\in\mathbb{N}} : \inf_{n\in\mathbb{N}} u(x_n - y_n) \geq \delta\right\}\right) = 1. \tag{12}$$

Proof Let $x > 0$. We construct $\sigma \in \Sigma$ in a such a way (12) to be satisfied. At the begin suppose $x < K(\underline{\omega})$. Then by Assumption 4, $q(x, \underline{\omega}) > x > q(0, \underline{\omega}) = 0$. By standard Darboux Theorem there is $y^*(x) \in (0, x)$ such that $q(y^*(x), \underline{\omega}) = x$. Define $\sigma_1(x_1) = y^*(x_1)$ and for each $n > 1$ and history $h := (x_n, y_n)_{n\in\mathbb{N}} \in H$, $\sigma_n(\cdot|h_n, x_n) \equiv y^*(x_1)$. We show that such σ is feasible. We show more:

$$P_x^\sigma\left(h = (x_n, y_n)_{n\in\mathbb{N}} : \inf_{n\in\mathbb{N}} x_n \geq x_1\right) = 1. \tag{13}$$

Since $x_1 < K(\underline{\omega})$, hence by Assumption 4 we have

$$x_2 = q(y^*(x_1), \omega_1) \geq q(y^*(x_1), \underline{\omega}) = x_1.$$

Suppose $x_k \geq x_1$ $P_x^\sigma-$ a.s. Then repeating the reasoning above $x_{k+1} \geq x_1$ as desired. Therefore, such σ is feasible and (13) holds, hence also (12) with $\delta = u(x_1 - y^*(x_1)) > 0$. Now assume $x_1 \geq K(\underline{\omega})$. Put any $x_0 \in (0, K(\underline{\omega}))$. Then let us define $\sigma_1(x_1) = y^*(x_0)$ and for $n > 1$, $\sigma_n(\cdot|h_{n-1}, x_n) \equiv y^*(x_0)$. Similarly as before we show that such σ is feasible and (12) holds, with $\delta = u(x_0 - y^*(x_0))$.

First main result is existence and global attracting property of utility function in (1) under additional assumption that u is bounded function.

Theorem 2 *Assume 1, 2, 3, 4, 5 and* $u \in B(X)$*. Then there exists unique recursive utility function such that* $U^* \in \mathbf{U}_+^o$*:*

(i) U^* has global attracting property on $\mathbf{U}_+^o$ i.e.

$$\lim_{n\to\infty} ||U^* - T_W^n(U)||_{X\times\Sigma} = 0$$

whenever $U \in \mathbf{U}_+^o$;

(ii) The truncation error satisfies:

$$||T_W^n(U) - U^*||_{X\times\Sigma} \leq M\left(1 - \alpha^{r^n}\right) \text{ for all } n \in \mathbb{N}, \tag{14}$$

whenever $U \in \mathbf{U}_+^o$. Here $M = 2||U||_{X\times\Sigma}$, $\alpha = \frac{t_0}{s_0}$ and t_0 and s_0 are choosen in a following way:

$$0 < t_0 < 1 < s_0 \text{ and it holds } t_0^{1-r}U(\cdot) \leq T_W(U)(\cdot) \leq s_0^r U(\cdot).$$

(iii) J is recursive utility function and $J(x,\sigma) = U^*(x,\sigma)$ for each $x > 0$, and $\sigma \in \Sigma$ satisfying

$$P_x^\sigma\left(h := (x_n, y_n)_{n\in\mathbb{N}} \inf_{n\in\mathbb{N}} : u(x_n - y_n) \geq \delta\right) = 1$$

for some $\delta > 0$.

Proof We prove (i) and (ii) together. Moreover, we prove analogous thesis for all operators T_W^δ $(\delta \geq 0)$. Let $\delta \geq 0$. Put $t \in (0,1)$, $U \in \mathbf{U}_+^o$ and $(x,\sigma) \in X \times \Sigma$. Then it holds

$$\begin{aligned} T_W^\delta(tU)(x,\sigma) &= \int_{\Gamma(x)} W\left(u_\delta(x-y), \mathcal{M}_y(tU(\cdot,\sigma^2(x,y)))\right)\sigma_1(dy|x) \\ &\geq \int_{\Gamma(x)} W\left(u_\delta(x-y), t\mathcal{M}_y(U(\cdot,\sigma^2(x,y)))\right)\sigma_1(dy|x) \end{aligned} \tag{15}$$

$$\geq t^r \int_{\Gamma(x)} W\left(u_\delta(x-y), \mathcal{M}_y(U(\cdot,\sigma^2(x,y)))\right)\sigma_1(dy|x) = t^r T_W^\delta(U)(x,\sigma). \tag{16}$$

Here (15) follows from subhomogenity of $\mathcal{M}_y$ (Assumption 5) and (16) follows from Assumptions 3. As a result, by Theorem 1 there is unique $U^\delta \in \mathbf{U}_+^o$ such that U^δ is a fixed point T_W^δ and (after replacing T_W be T_W^δ) satisfies (i) and and (ii) of this theorem. In particularly, we put $U^* := U^0$ which satisfies (i) and (ii).
We prove (iii). For $n \in \mathbb{N}$ let us define $J_n^\delta(x,\sigma) := (T_W^\delta)^n(\mathbf{0})(x,\sigma)$ (n-th composition of $\mathbf{0}$ - function). By Lemma 2, $J_n^\delta \in \mathbf{U}_+^o$ whenever $\delta > 0$. By Theorem 1

$$\lim_{n\to\infty} ||J_n^\delta - U^\delta||_{X\times\Sigma} = 0 \tag{17}$$

and U^δ and satisfies (i) and (ii) of this theorem.

By Lemma 2 (iv) $J \leq U^*$, hence J is always finite. We show opposite inequality. For $x > 0$ and $\delta > 0$, let us define $\Sigma_{x,\delta} \subset \Sigma$ in such a way: $\sigma \in \Sigma_{x,\delta}$ if and only if

$$P_x^\sigma\left(\left\{h = (x_n, y_n)_{n\in\mathbb{N}} \in H : \inf_{n\in\mathbb{N}} u(x_n - y_n) \geq \delta\right\}\right) = 1. \tag{18}$$

Observe that by Lemma 3, $\Sigma_{x,\delta}$ is nonempty. Put any $\sigma \in \Sigma$.
Let $Q(\cdot|y) := \rho q_y^{-1}(\cdot)$.

We show that $J_n(x,\sigma) = (T_W^\delta)^n(\mathbf{0})(x,\sigma)$ for $n \geq 2$. By definition of J_1, u_δ and (18), we have

$$J_1(x,\sigma) = \int_{\Gamma(x)} W(u(x-y),0)\sigma_1(dy|x) = \int_{\Gamma(x)} W(u_\delta(x-y),0)\sigma_1(dy|x) = T_W^\delta(\mathbf{0})(x,\sigma).$$

Suppose that

$$J_k(x',\sigma') = (T_W^\delta)^k(\mathbf{0})(x',\sigma') \tag{19}$$

for some integer k and each $(x',\sigma') \in X \times \Sigma_{x',\delta}$. Since $\sigma \in \Sigma_{x,\delta}$, hence $\sigma(x,y) \in \Sigma_{x',\delta}$ for π_2- a.a. $(x,y,x') \in H_2 \times X$, where π_2 is a marginal of P_x^σ on $H_2 \times X$. Hence, by (19) we have

$$J_{k+1}(x,\sigma) = \int_{\Gamma(x)} W(u(x-y), \mathcal{M}_y(J_k(\cdot,\sigma^2(x,y))))\sigma_1(dy|x) = (T_W^\delta)^{k+1}(\mathbf{0})(x,\sigma). \tag{20}$$

As a result for each k, $J_k(x,\sigma) = (T_W^\delta)^k(\mathbf{0})(x,\sigma)$. Hence and by (17), $J(x,\sigma)$ coincides with U^δ for such (x,σ). As a result $J(x,\sigma) \geq U^*(x,\sigma)$.

In the next theorem, we relax the requirement u to be bounded on X. The point is u is bounded on all compact subintervals of X, since it is continuous function. Moreover, Assumption 4 prevents extension of some fixed level of capital regardless of realization of the noise.

For each interval $I \subset X$ let us consider restriction of the model $(X, \Gamma, \Omega, u, q, W, \mathcal{M})$ to that model, where X has been replaced by I, whenever I is *invariant* i.e. satisfies the following condition:

$$x \in I \text{ implies that for each } y \in \Gamma(x), \omega \in \Omega \text{ it holds } q(y,\omega) \in I.$$

The aforementioned condition means that transition probability generated by production function q moves all states from I into perhaps another state x' but still $x' \in I$. Let $\mathcal{I}$ be a family of invariant sets.

Lemma 4 *Assume 4. Then*

$$\{[0,\xi] : \xi \geq K(\overline{\omega})\} \subset \mathcal{I}.$$

Proof Suppose $\xi \geq K(\overline{\omega})$. Then if $x \leq \xi$ then for each $y \in \Gamma(x)$, $\omega \in \Omega$, it holds (from Assumption 4)

$$q(y,\omega) \leq q(\xi,\overline{\omega}) \leq \xi.$$

By Lemma 4 all sets on the form $[0,\xi]$, where $\xi \geq K(\overline{\omega})$ are invariant. Hence if $h = (x_n, y_n)_{n\in\mathbb{N}}$ is a history generated by σ, one must happen: $\sup_{n\in\mathbb{N}} x_n \leq \max(K(\overline{\omega}), x_1)$.

For all $\xi \geq K(\overline{\omega})$ let us define $I_\xi := [0,\xi]$. For all ξ let us consider the models $(I_\xi, \Gamma, \Omega, u, q, W, \mathcal{M})$, where u and $q(\cdot,\omega)$ are restricted to I_ξ. Clearly u is bounded on I_ξ.

Let Σ_ξ be a set of policies from Σ, where initial, and hence all states are restricted to I_ξ. Put $\mathbf{U}_+^\xi$ as a set of all $U \in \mathbf{U}_+$ restricted to $I_\xi \times \Sigma_\xi$ and $(\mathbf{U}_+^\xi)^o$ set of all elements of $\mathbf{U}_+$ having strictly positive infimum on $I_\xi \times \Sigma_\xi$.

Let

$$\mathbf{U}_0 := \bigcap_{\xi > 0} (\mathbf{U}_+^\xi)^o.$$

Theorem 3 *Assume 1, 2, 3, 4 and 5. There exist unique recursive utility function such that $U^* \in \mathbf{U}_0$ such that*

(i) For each $\xi > 0$, $U \in \mathbf{U}_0$, U^* satisfies

$$\lim_{n\to\infty} ||U^* - T_W^n(U)||_{I_\xi \times \Sigma_\xi} = 0;$$

(ii) For each $\xi > 0$ the truncation error satisfies:

$$||T_W^n(U) - U^*||_{I_\xi \times \Sigma_\xi} \leq M\left(1 - \alpha^{r^n}\right) \text{ for all } n \in \mathbb{N}. \tag{21}$$

Here $M = 2||U||_{I_\xi \times \Sigma_\xi}$, $\alpha = \frac{t_0}{s_0}$ and t_0 and s_0 are chosen in a following way:

$$0 < t_0 < 1 < s_0 \text{ and it holds } t_0^{1-r} U(\cdot) \leq T_W(U)(\cdot) \leq s_0^r U(\cdot).$$

(iii) J is recursive utility function and $J(x,\sigma) = U^*(x,\sigma)$ for all $x > 0$ and $\sigma \in \Sigma$ such that

$$P_x^\sigma \left(h := (x_n, y_n)_{n\in\mathbb{N}} \inf_{n\in\mathbb{N}} u(x_n - y_n) \geq \delta \right) = 1$$

for some $\delta > 0$.

Proof By Theorem 2 for each $\xi > 0$ there are $U^\xi \in (\mathbf{U}_+^\xi)_+$ satisfying all condition (i)-(iv) of this theorem. We show that there is $U^* \in \mathbf{U}_0$ such that $U^*|_{I_\xi \times \Sigma_\xi} \equiv U^\xi$. Let us define $U^*(x,\sigma) := U^\xi(x,\sigma)$ whenever, $(x,\sigma) \in I_\xi \times \Sigma_\xi$. We show that this U^* is well defined i.e. if $\xi' < \xi$ then $U^{\xi'} = U^\xi|_{I_{\xi'} \times \Sigma_{\xi'}}$. Observe that, since $U^\xi \in (\mathbf{U}_+^\xi)^o$ and $I_{\xi'} \times \Sigma_{\xi'} \subset I_\xi \times \Sigma_\xi$, hence

$$0 < \inf_{(x,\sigma)\in I_{\xi'} \times \Sigma_{\xi'}} U^\xi(x,\sigma) \leq \sup_{(x,\sigma)\in I_{\xi'} \times \Sigma_{\xi'}} U^\xi(x,\sigma) < \infty. \tag{22}$$

In particular for all $(x,\sigma) \in I_{\xi'} \times \Sigma_{\xi'}$ it holds

$$U^\xi(x,\sigma) = T(U^\xi)(x,\sigma). \tag{23}$$

Since $U^{\xi'}$ is unique function satisfying (22) and (23) hence $U^{\xi'} = U^\xi|_{I_{\xi'} \times \Sigma_{\xi'}}$. As a result U^* is well defined function and satisfies all thesis of this theorem.

5 Bellman equations and existence of optimal program

In this section, we shall study an optimization problem and find

$$v^*(x) = \sup_{\sigma\in\Sigma} J(x,\sigma)$$

and policy $\sigma^* \in \Sigma$ realizing this supremum. Similarly as in standard dynamic programing, we construct Bellman equation i.e.

$$v^*(x) = \sup_{y\in\Gamma(x)} W(u(x-y), \mathcal{M}_y(v^*))$$

and examine existence and desired properties of the stationary optimal policy as an argmax correspondence in Bellman equations.

Put

$$BP(v)(x) = \sup_{y\in\Gamma(x)} W(u(x-y), \mathcal{M}_y(v)).$$

In the remaining part of this paper we slightly abuse notation and we will denote $B := B_+(X)$ and $B^o := (B_+(X))^o$.

Theorem 4 *Under Assumption 1, 2, 3, 4, 5 and $u \in B(X)$, BP maps B_+^o into itself and*

(i) there exists unique fixed point v^* of BP such that $v^* \in B_+^o$;
(ii) for each $v \in B_+^o$

$$\lim_{n\to\infty} ||BP^n(v) - v^*||_X = 0$$

and the truncation error satisfies:

$$||BP^n(v) - v^*||_X \leq M\left(1 - \alpha^{r^n}\right) \text{ for all } n \in \mathbb{N}. \tag{24}$$

Here $M = 2||v||_X$, $\alpha = \frac{t_0}{s_0}$ and t_0 and s_0 are choosen in a following way:

$$0 < t_0 < 1 < s_0 \text{ and it holds } t_0^{1-r} v(\cdot) \leq BP(v)(\cdot) \leq s_0^r v(\cdot);$$

(iii) v^* *is increasing, continuous and satisfies*

$$v^*(x) = \sup_{\sigma \in \Sigma} J(x, \sigma),$$

for each $x > 0$*;*

(iv) There exists pure and stationary optimal policy σ^* *satisfying:*

$$W(x - \sigma^*(x), \mathcal{M}_{\sigma^*(x)}(v^*)) = \max_{y \in [0,x]} W(u(x-y), \mathcal{M}_y(v^*)) \quad (25)$$
$$= \max_{y \in [0,x]} W(u(x-y), \mathcal{M}_y(\tilde{v})),$$

where $\tilde{v}(x) = \sup_{\sigma \in \Sigma} J(x, \sigma)$ *for* $x > 0$ *and* $\tilde{v}(x) = 0$*. Moreover,* $J(x, \sigma^*) = \tilde{v}(x)$ *for all* $x \in X$*.*

Proof We show (i) and (ii) holds. First we show that BP maps B^o_+ into itself. Indeed, if $v \in B^o_+$ then $||v||_X < \infty$, $[v]_X > 0$ and for each $y \in \Gamma(x)$

$$BP(v)(x) \geq W(u(x,y), \mathcal{M}(v)(x,y)) \geq W(0, [v]_X) > 0.$$

Moreover,

$$BP(v) \leq W\left(||u||_X, ||v||_X\right) < \infty.$$

To show that $BP(v)(\cdot)$ is measurable, observe that by Assumptions 2 $BP(v)(x)$ must be increasing in x, regardless on v. Hence $BP(v)(\cdot)$ is Borel measurable. Therefore, $BP(v) \in B^o_+$. As a result BP maps B^o_+ into itself.

By Assumption 3 and 5 it can be easily concluded that $BP(tv)(\cdot) \geq t^r BP(v)(\cdot)$ for each $v \in B^o_+$ and $t \in (0,1)$. As a result by Theorem 1 (i) and (ii) holds.

We show (iii). It is easy to see v^* is increasing. Let $\sigma \in \Sigma$. We show that $v^*(x) \geq J(x,\sigma)$ or equivalently $v^*(x) \geq J_n(x,\sigma)$ for each $n \in \mathbb{N}$. For each n let $\sigma \in \Sigma$ be arbitrary. For each $x \in X$ it holds

$$v^*(x) = \sup_{y \in \Gamma(x)} W(u(x,y), \mathcal{M}_y(v^*)) \geq \int_{\Gamma(x)} W(u(x,y_1), \mathcal{M}_{y_1}(v^*))\sigma_1(dy_1|x).$$

Consequently, for $\tau \leq n$ and history $h := (x_n, y_n)_{n \in \mathbb{N}}$

$$v^*(x_\tau) \geq \int_{\Gamma(x_\tau)} W(u(x_\tau, y_\tau), \mathcal{M}_{y_\tau}(v^*))\sigma_\tau(dy_\tau|h_\tau, x_\tau).$$

Hence, by Lemma 2 $v^*(x) \geq T^n_W(v^*)(x,\sigma) \geq T^n_W(\mathbf{0})(x,\sigma) = J_n(x,\sigma)$. As a result $v^*(x) \geq J(x,\sigma)$ and $v^*(x) \geq \sup_{\sigma \in \Sigma} J(x,\sigma)$. To finish the proof, we need to show the equality for $x > 0$. Put $\tilde{v}(x) = \sup_{\sigma \in \Sigma} J(x,\sigma)$ for $x > 0$ and $\tilde{v}(0) = 0$. We show that $\tilde{v}$ is fixed point of BP. For $x = 0$ this theses becomes trivial. Put $x > 0$, $y \in [0,x]$ and consider a policy $\tilde{\sigma} \in \Sigma$ defined as follows: $\tilde{\sigma}_1(x) = y$ and for $k > 1$ let $\tilde{\sigma}_k := \sigma_{k-1}$, where $\sigma \in \Sigma$ is chosen arbitrarily. Then by definition of $\tilde{v}$ and $\tilde{\sigma}$ it holds

$$\tilde{v}(x) \geq W(u(x-y), \mathcal{M}_y(J(\cdot,\sigma)))$$

By Assumptions 3, 5 and above

$$\tilde{v}(x) \geq \sup_{y \in \Gamma(x)} W(u(x-y), \mathcal{M}_y(\tilde{v})) = BP(\tilde{v})(x), \quad (26)$$

since y is chosen arbitrarily. To show that $\tilde{v}$ satisfies opposite inequality, observe that for each $x > 0$ and $\sigma \in \Sigma$ it holds

$$J(x,\sigma) \leq \sup_{y\in\Gamma(x)} W(u(x-y),\mathcal{M}_y(J(\cdot,\sigma^2(x,y))))$$
$$\leq \sup_{y\in\Gamma(x)} W(u(x-y),\mathcal{M}_y(\tilde{v}))) = BP(\tilde{v})(x),$$

hence taking a supremum over $\sigma \in \Sigma$ it holds $\tilde{v}(x) \leq BP(\tilde{v})(x)$. Together with (26) $\tilde{v}$ is fixed point of BP.

Combining Lemma 3 and Theorem 2

$$\inf_{x\in X\setminus\{0\}} \tilde{v}(x) = \inf_{x\in X\setminus\{0\}} \sup_{\sigma\in\Sigma} J(x,\sigma) > 0.$$

As a result a function $v^{**}(x) := \tilde{v}(x)$ for $x > 0$ and $v^{**}(0) = v^*(0)$ is in the interior of B_+. We show that it is fixed point of BP. By Assumption 5 $\mathcal{M}_y(\tilde{v}) = \mathcal{M}_y(v^{**})$, for all $y > 0$. Hence for such y $W(u(x-y),\mathcal{M}_y(\tilde{v})) = W(u(x-y),\mathcal{M}_y(v^{**}))$. As a result

$$\sup_{y\in(0,x]} W(u(x-y),\mathcal{M}_y(v^*)) = \sup_{y\in(0,x]} W(u(x-y),\mathcal{M}_y(v^{**})).$$

On the other hand

$$W(u(x-0),\mathcal{M}_0(v^*)) = W(u(x-0),v^*(0)) = W(u(x-0),v^{**}(0)) = W(u(x-0),\mathcal{M}_0(v^{**})),$$

hence $v^{**} = BP(v^*) = BP(v^{**})$, hence v^{**} is fixed point of BP. Since by part (i) of this proof v^* is unique fixed point of BP in B_+^o, hence $v^* \equiv v^{**}$. As a result $\tilde{v}(x) = v^*(x)$ for $x > 0$.

Finally we show that $\tilde{v}$ is continuous on $X \setminus \{0\}$. Before we examine a continuity of v^* on X. Clearly, unit constant is continuous, and by proven part (ii) $v^*(\cdot) = BP^n(\mathbf{1})(\cdot)$. Moreover, this limit is uniform. Hence, we simply need to show show that $BP(v)$ is continuous function whenever v is continuous and $[v]_X > 0$. Let v be such function. Then by Assumption 5 $\mathcal{M}_y(v)$ is continuous in y, hence $W(u(x-y),\mathcal{M}_y(v))$ is jointly continuous. As a result by Berge Maximum Theorem (Theorem 17.31 in [1]) $BR(v)(\cdot)$ must be continuous and consequently v^* is continuous, hence $\tilde{v}$ is continuous on $X \setminus \{0\}$.

Proof of (iv). From (iii) v^* is continuous on $X \setminus \{0\}$, by Assumptions 2, 3 and 5 there is σ^* satisfying (25). By (iii) for $x > 0$, on can be obtain $v^*(x) = \tilde{v}(x) = J(x,\sigma^*)$.

Let us relax assumption $u \in B(X)$ in the final main theorem. Then following Theorem 5 is simple consequence of Theorem 4.

Theorem 5 *Under Assumptions 1, 2, 3, 4 and 5 , BP maps $B(I_\xi)_+^o$ into itself for all $\xi > 0$ and*

(i) there exists unique fixed point v^* of BP such that $v^* \in B_+^o(I_\xi)$ for all $\xi > 0$;

(ii) for each $v \in B_+^o(I_\xi)$

$$\lim_{n\to\infty} ||BP^n(v) - v^*||_{I_\xi} = 0$$

and the truncation error satisfies:

$$||BP^n(v) - v^*||_{I_\xi} \leq M\left(1-\alpha^{r^n}\right) \text{ for all } n \in \mathbb{N}.$$

Here $M = 2||v||_{I_\xi}$, $\alpha = \frac{t_0}{s_0}$ and t_0 and s_0 are choosen in a following way:

$$0 < t_0 < 1 < s_0 \text{ and it holds } t_0^{1-r}v(\cdot) \leq BP(v)(\cdot) \leq s_0^r v(\cdot);$$

(iii) v^* satisfies

$$v^*(x) = \sup_{\sigma\in\Sigma_\xi} J(x,\sigma),$$

for each $x > 0$;

We omit the proof since the argument is the same along the lines as in Theorem 3.

6 Problems

6.1 Deterministic model with general discounting function

Consider an example of the model satisfying 2, 3, 5 but the Assumption 4 is relaxed. In this section it is constructed a model in which an optimal policy does not exist. Consider deterministic model in which aggregator W is on the form $W(v_1, v_2) = v_1 + \beta\sqrt{v_2}$ with $\beta \in (0,1)$, instantaneous utility is $u(x) = x$ and the production function is deterministic $q(y) = y$. Comparing this model with [30] this is a model with discounting function in a form $\delta(x) = \beta\sqrt{x}$, but observe that $\delta'(0) = \infty$. We show that that there is no optimal investment program $(y_n^*)_{n\in\mathbb{N}}$. For each $x \geq 0$ let us define

$$S(x) := \left\{ (y_n)_{n\in\mathbb{N}} \in \mathbb{R}_+^\infty : \sum_{n=1}^{\infty} y_n = x \right\}.$$

Put $v^*(x) := \sup_{\sigma\in\Sigma} J(x,\sigma)$. Observe that maximization problem of $J(x,\cdot)$ is equivalent to find a sequence $(y_n^*)_{n\in\mathbb{N}}$ maximizing

$$\lim_{n\to\infty} \left(y_1 + \beta\sqrt{y_2 + \beta\sqrt{y_3 + \ldots + \beta\sqrt{y_n}}} \right),$$

such that $(y_n)_{n\in\mathbb{N}} \in S(x)$.

Let $v_0(x) = 0$, and for $n > 1$

$$v_{n+1}(x) = \sup_{y\in\Gamma(x)} \left(x - y + \beta\sqrt{v_n(y)} \right).$$

It is easy to verify $v_1(x) = x$ and for $n \geq 2$

$$v_n(x) = \begin{cases} \beta^{2-2^{1-n}} x^{2^{-n}} & \text{if } x \leq \beta^2/\phi_{n-1} \\ x - \beta^2/\phi_{n-1} + \beta^2/\psi_{n-1} & \text{if } x > \beta^2/\phi_{n-1} \end{cases},$$

where $\phi_n := 2^{\frac{n+1}{1-\frac{1}{2^{n+1}}}}$, $\psi_n := 2^{\frac{n+1}{2^n-\frac{1}{2}}}$ $(n = 1,2,\ldots)$. Obviously, $\phi_n \to \infty$ and $\psi_n \to 1$ as $n \to \infty$. Hence for each $x \in X$, $v_n(x) \to \tilde{v}(x)$, where $\tilde{v}(x) = x + \beta^2$ for all $x > 0$ and $\tilde{v}(0) = 0$. Put $v^*(x) = x + \beta^2$ for all $x \geq 0$. Observe that $\tilde{v}$ and v^* are both fixed points of BP. Repeating the reasoning in Theorem 4 one can be proven, there is unique fixed point of BP in $B_+^o(X)$, $\tilde{v}$ is optimal value function and (25) is satisfied. We show that there exist no optimal program. On the contrary suppose there exist optimal $(y_n^*)_{n\in\mathbb{N}}$. Since it must hold

$$\tilde{v}(x) = \max_{y\in[0,x]} (x - y + \beta\sqrt{y + \beta^2}),$$

hence $y_1^* = 0$. Consequently $y_2^* = y_3^* = \ldots = 0$. But this program gives x instead of $x + \beta^2$. Hence no program is optimal.

6.2 Discounting utility functions

Observe that, if W has affine form and $\mathcal{M}_y$ is expectation then the model in this paper reduces to the standard discounting growth model. But in this model neither Assumptions 5 nor 3 are satisfied. Observe however that fixed point does not belong to interior i.e. $[v^*]_X = 0$. As example consider deterministic growth model with $u(x) = \sqrt{x}$, $\beta \in (0,1)$ and $q(y) = \sqrt{y}$. Then Bellman equation has a form $BP(v) = \sup_{y\in[0,x]} \left(\sqrt{x-y} + \beta\sqrt{v(y)} \right)$. Then optimal value function has $v^*(x) = \lim_{n\to\infty} BP^n(\mathbf{0})$. Observe that $v_n(x) = \sqrt{1+\beta^2 m_n^2}\sqrt{x}$, where $m_1 = 0$ and for $n > 1$ $m_{n+1} = \sqrt{1+\beta^2 m_n^2}$. It is easy to verify $\lim_{n\to\infty} m_n = \frac{1}{1-\beta^2}$, hence $v^*(x) = \sqrt{1 + \frac{\beta^2}{1-\beta}}\sqrt{x}$ and $\inf_{x>0} v^*(x) = 0$. The same result can be obtain by limit of $BP^n(\mathbf{1})$, hence neither Theorem 4 nor 5 is satisfied. For other problems with uniqueness of fixed point of BP in discounting model the reader is reffered to Kamihigashi [32].

6.3 Asymptotic properties of recursive utilities with varied instantaneous utilities.

Assumption 3 excludes many interesting economic models. For example excludes affine aggregator, and consequently the standard $\beta-$discounting model, as well as risk sensitive model as in Bäuerle and Jaśkiewicz [6], logarithmic aggregator in Koopmans et all in [34] as well as useful Thompson aggregator in a form

$$W(v_1, v_2) = \left(v_1^{\xi} + \beta v_2^{\eta}\right)^{\frac{1}{\eta}},$$

with $\xi > 0$, $\eta > 1$ and $\beta \in (0, 1)$.

Example 2 Suppose that instantenous utility u is bounded from from bellow by strictly positive value δ and from above by some finite value. Consider aggregator $W(v_1, v_2) = v_1 + \beta v_2$. Observe that $J(x, \sigma) \leq M := \frac{||u||_X}{1-\beta}$ for all $(x, \sigma) \in X \times \Sigma$. Then aggregator W is equivalent $\tilde{W}(v_1, \min(M, v_1))$ i.e. J in (1) is the same if we put $\tilde{W}$ instead of W. Then we will find $r \in (0, 1)$ such that for all $t \in (0, 1)$, $(v_1, v_2) \in \mathbb{R}_+^2$

$$\frac{\tilde{W}(v_1, tv_2)}{\tilde{W}(v_1, v_2)} = \frac{v_1 + \beta \min(M, tv_2)}{v_1 + \beta \min(M, v_2)} \geq t^r. \tag{27}$$

Observe that $tv_2 \leq v_2$. If $M \leq tv_2 \leq v_2$ we have

$$\frac{\tilde{W}(v_1, tv_2)}{\tilde{W}(v_1, v_2)} = 1.$$

If $tv_2 \leq M \leq v_2$ we have

$$\frac{\tilde{W}(v_1, tv_2)}{\tilde{W}(v_1, v_2)} = \frac{v_1 + \beta tv_2}{v_1 + \beta M} \geq \frac{v_1 + \beta tM}{v_1 + \beta M}.$$

If $v_2 \leq M$ then

$$\frac{\tilde{W}(v_1, tv_2)}{\tilde{W}(v_1, v_2)} = \frac{v_1 + \beta tv_2}{v_1 + \beta M} \geq \frac{v_1 + \beta tv_2}{v_1 + \beta v_2}.$$

Note that,

$$pt + (1 - p) \geq t^r$$

for all $t \in (0, 1)$ whenever $0 < p < r$. Hence if we put $r := \frac{\beta M}{\delta + \beta M}$ then (27) holds. We can consider similar procedure along the lines for aggregator $W(v_1, v_2) = \frac{1}{\theta} \ln\left(1 + v_1^{\xi} + \beta v_2\right)$ for $\xi > 0$ and $W(v_1, v_2) = (1 + v_1 + \beta v_2^{\eta})^{\frac{1}{\eta}}$, for $\eta > 1$ and $\theta > 0$.

By the above example we may consider asymptotic properties of optimal policy and value with affine and logarithmic aggregator also and we may consider daily utility at the form $u_\delta(x) = \max(u(x), \delta)$. Then we may take a limit with $\delta \to 0^+$ and consider approach of optimal value by optimal J whenever u is changed by u_δ.

7 Concluding remarks

This paper consider existence and uniqueness of recursive utility with nonlinear CES and aggregator. Moreover, it is considered optimization problem by Bellman equations. For instance it is assumed law of motion is like in growth model. This model in some sens extend general discounting models due to Jaśkiewicz et all [30], [31]. Some Thompson aggregators (according Marinacci and Montrucchio [40] terminology) belong to this class. But this paper contains an example of aggregator do not belonging to Thompsons collection but satisfying assumptions of this paper. Also CES is nonlinear, hence this paper extends deterministic models like Bich et all. [9], Felipe da Rocha and Vailakis [22] or Le Van and Vailakis [36]. Observe, however (unlike Bich et all [9] and Nowak and Matkowski [42] or Jaśkiewicz et all [30], Le

Van and Vailakis [36]) Assumption 3 requires that immediate return u to be bounded from bellow. To find existence recursive utility and its optimal strategy an iterative algorithm on solid cones is proposed. It is used Guo et. all stuff [25] as an alternative for standard Banach Contraction Principle with standard norm or with Thompson metric (e.g. in [40] or in [22]) and its extension due to Matkowski [41] (e.g. in [30]). The possibility of an application of various discount functions allows to take into account different aspects of discounting discussed in the area of finance, economics, psychology, environmental management (e.g. Frederic et. all [23], and Green et. all [24]).

On the other hand, there were not moved on another interesting open problems, solved standard in dynamic programming, such as stability of optimal paths, differentiability of a value function, Euler equations etc. and all of them are left as an open problem.

8 Appendix

Lemma 5 *For each $X \subset \mathbb{R}$*

$$B_+^o(X) = \{v \in B_+(X) : [v]_X > 0\}$$

Proof Let $v \in B_+^o(X)$. Then there is $\epsilon > 0$, such that for all ϕ such that $||\phi||_X \leq 1$ the function $v(\cdot) - \epsilon\phi(\cdot) \in B_+(X)$. In particularly it holds for $\phi \equiv \mathbf{1}$, hence for each $x \in X$ $v(x) - \epsilon \geq 0$, hence $[v]_X > \epsilon > 0$.

Conversely, suppose $[v]_X > 0$. Then if $||\phi||_X = 1$ then $v(x) - \phi(x)[v]_X \geq 0$ for all $x \in X$. Indeed,

$$v(x) - \phi(x)[v]_X \geq [v]_X(1 - \phi(x)) \geq 0$$

for all $x \in X$, hence the ball in $B(X)$ centered in v and radius $[v]_X$ is included in $B_+(X)$. Hence $v \in B_+^o(X)$.

References

1. Aliprantis, C.D. and Border, K.C. (2006). Infinite Dimensional Analysis. Springer-Verlag Berlin Heidelberg,
2. Amir, R. (1997). A new look at optimal growth under uncertainty. Journal of Economic Dynamics and Control, vol. 22, pp. 67-86
3. Balbus, Ł., Jaśkiewicz, A. and Nowak, A.S. (2015). Existence of Stationary Markov Perfect Equilibria in Stochastic Altruistic Growth Economies. Journal of Optimization Theory and Applications, vol. 165, no 1, pp 295-315.
4. Balbus, Ł., Reffett, K. and Woźny, Ł. (2012). Stationary Markovian equilibrium in altruistic stochastic OLG models with limited commitment. Journal of Mathematical Economics Volume, vol. 48 (2), pp. 115–132.
5. Balbus, Ł., Reffett, K. and Woźny, Ł. (2013). A constructive geometrical approach to the uniqueness of Markov stationary equilibrium in stochastic games of intergenerational altruism. Journal of Economic Dynamics & Control, vol. 37, pp. 1019–1039.
6. Bäuerle, N., Jaskiewicz, A. (2015). Stochastic Optimal Growth Model with Risk Sensitive Preferences. Cornell University Library.
7. Becker, R. A., Boyd, J. H. III (1997). Capital theory, equilibrium analysis and recursive utility. NewYork: Blackwell.
8. Bernheim, D.A., Ray, D. (1989). Markov perfect equilibria in altruistic growth economies with production uncertainty. Journal of Economic Theory, vol. 47, pp. 195-202.
9. Bich, P., Drugeon, J-P. and Morhaim, L. (2015). On aggregators and dynamic programming. Paris School of Economics.
10. Blackwell, D. (1965). Discounted dynamic programming. Annals of Mathematical Statistics, vol. 36, pp 226-235.
11. Boyd, J.H. III (2006). Discrete-Time Recursive Utility. Discrete-time recursive utility. R. A. Dana, C. Le Van, T. Mitra, K. Nishimura (Eds.). Handbook of optimal growth 1, discrete time, pp. 251272, New York: Springer.
12. Brock, W.A., Mirman, L.J. (1972). Optimal Economic Growth And Uncertainty: The Discounted Case. Journal of Economic Theory, vol. 4, pp. 479-533.
13. Chew, S.H. (1983) A generalization of the quasilinear mean with applications to the measurement of income inequality and decision theory resolving the Allais Paradox. Econometrica, vol. 51, no. 4, pp. 1065-1092.
14. Chew. S.H., Epstein, L.G (1989) The structure of preferences and attitudes toward the timing of the resolution of uncertainty. International Economic Review, vol. 30, no. 103–117.
15. Dekel, E. (1986). An axiomatic characterization of preferences under uncertainty: weakening of independence axiom. Journal of Economic Theory, vol 40. pp. 304-318.
16. Denardo, E.V. (1967). Contraction Mappings in the Theory Underlying Dynamic Programming. SIAM Review, vol. 9 (2), pp. 165-177.

17. Dugundji, J., Granas, A. (2003). Fixed Point Theory. Springer Monographs in Mathematics, Springer, New York, 2003.
18. Dutta, P., Sundaram, R. (1992). Markovian equilibrium in a class of stochastic games: existence theorems for discounted and undiscounted models. Economic Theory, vol. 2, pp. 197-214.
19. Epstein, L.G., Zin, S.E. (1989). Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: a theoretical framework. Econometrica, vol. 57, no. 4, pp. 937-969.
20. Feinberg, E. A. (2002). Total reward criteria. E. A. Feinberg, A. Shwartz (Eds.), Handbook of Markov decision processes, pp. 173207, Boston: Kluwer.
21. Gul, F. (1991). A theory of disappointment aversion. Econometrica, vol. 59, no 3, pp. 667-686.
22. Filipe Martins-da-Rocha, V., Vailakis, Y. (2010). Existence and uniqueness of fixed-point for local contractions. Econometrica, vol. 78, no. 3, pp. 11271141.
23. Frederick, S., Loewenstein, G., and O'Donoghue, T. (2002). Time discounting and time preference: a critical review. Journal of Economic Literature, vol. 11, pp. 351401.
24. Green, L., Myerson, J., and Mc Fadden, E. (1997). Rate of temporal discounting decreases with amount of reward. Memory & Cognition, vol. 25, pp. 715723.
25. Guo, D., Cho, Y.J. and Zhu, J. (2004). Partial Ordering Methods in Nonlinear Problems. Nova Science Publishers, Inc. New York.
26. Hansen, L.P., Sargent, T.J. (1995). Discounted linear exponential quadratic Gaussian control. IEEE Trans. Automat. Control. vol. 40, pp. 968-971.
27. Hernández-Lerma, O., and Lasserre, J. B. (1996). Discrete-time Markov control processes: basic optimality criteria. Springer New York.
28. Hernández-Lerma, O., and Lasserre, J. B. (1999). Further topics on discrete-time Markov control processes. Springer Ney York.
29. Jaśkiewicz, J. and Nowak, A.S. (2014). Stationary Markov perfect equilibria in risk sensitive stochastic overlapping generations models. Journal of Economic Theory, vol. 151. pp. 411-447.
30. Jaśkiewicz, A., Matkowski, J. and Nowak, A.S. (2014) On variable discounting in dynamic programming: applications to resource extraction and other economic models. Annals of Operations Research, vol. 220, pp. 263278.
31. Jaśkiewicz, A., Matkowski, J. and Nowak, A.S. (2015) Persistently Optimal Policies in Stochastic Dynamic Programming with Generalized Discounting. Mathematics of Operations Research, in print. http://dx.doi.org/10.1287/moor.1120.0561.
32. Kamihigashi, T. (2015) An Order Theoretic Approach to Dynamic Programming: An Exposition. Kobe University.
33. Koopmans, T. C. (1960). Stationary ordinal utility and impatience. Econometrica, vol. 28, pp. 287309.
34. Koopmans, T.C., Diamond, P.A. and Williamson, R.E. Stationary utility and time perspective. Econometrica, vol. 32, pp. 82100.
35. Kreps, D.M. and Porteus, E.L. (1978) Temporal Resolution of Uncertainty and Dynamic Choice Theory. Econometrica, vol. 46, no. 1, pp. 185-200.
36. Le Van, C. and Vailakis, Y. (2005) Recursive utility and optimal growth with bounded or unbounded returns. Journal of Economic Theory, vol. 123, pp. 187 - 209.
37. Lucas, R. E., and Stokey, N.L. (1984). Optimal growth with many consumers. Journal of Economic Theory, vol 32, pp 139-171.
38. Lucas, R. E., Stokey, N. L. and Prescott, E. (1989). Recursive methods in economic dynamics. Cambridge: Harvard University Press.
39. Majumdar, M., Sundaram, R. (1991). Symmetric Stochastic Games of Resource Extraction: The Existence of Non-Randomized Stationary Equilibrium. "Stochastic games and related topics", T. Raghavan, T. Ferguson, T. Parthasarathy, and O. Vrieez eds. Kluwer, Dordrecht.
40. Marinacci, M., and Montrucchio, L. (2010). Unique solutions for stochastic recursive utilities. Journal of Economic Theory, 145, 17761804.
41. Matkowski, J. (1975). Integral solutions of functional equations. Dissertationes Mathematicae, vol. 127, pp. 168.
42. Matkowski, J., Nowak, A.S. (2011). On Discounted Dynamic Programming with Unbounded Returns. Economic Theory, vol. 46, pp. 455-474.
43. Montrucchio, L. (1998). Thompson metric, contraction property and differentiability of policy functions. Journal of Economic Behavior & Organization, vol. 33, pp. 449-466.
44. Neveu, J. (1965). Mathematical Foundations of the Calculus of Probability. Holden-Day, San Francisco.
45. Ramsey, F. P. (1928). A mathematical theory of saving. Econometrics Journal, vol. 38, pp. 543599.
46. Rincón-Zapatero, J. P., and Rodriguez-Palmero, C. (2003). Existence and uniqueness of solutions to the Bellman equation in the unbounded case. Econometrica, vol. 71, pp. 1519-1555.
47. Rincón-Zapatero, J. P., and Rodriguez-Palmero, C. (2009). Corrigendum to "Existence and uniqueness of solutions to the Bellman equation in the unbounded case. Econometrica, vol. 71, pp. 1519-1555". Econometrica, vol. 77, pp. 317-318.
48. Samuelson, P. (1937). A note on measurement of utility. Review of Economic Studies vol. 4, pp. 155161.
49. Strauch, R. (1966). Negative dynamic programming. Annals of Mathematical Statistics, vol 37, pp. 871890.
50. Sundaram, R. (1989). Perfect Equilibrium in Non-randomized Strategies in a Class of Symmetric Dynamic Games. Journal of Economic Theory, vol. 47, pp. 153-177.
51. Perfect Equilibrium in Non-randomized Strategies in a Class of Symmetric Dynamic Games. Corrigendum. Journal of Economic Theory, vol. 49, pp. 385-387

52. Tarski, A. (1955). A lattice theoretical fixpoint theorem and its application. Pacific Journal of Mathematics, vol.5, pp. 285-309.
53. Thompson, A.C., (1963). On certain contraction mappings in a partially ordered vector space. Proceeding of American Mathematical Society, vol. 14, pp. 438-434.
54. Weil, P. (1993). Preauctionary savings and the permanent hypothesis. The Reviev of Economic Studies, vol. 60, no. 2, pp. 367–383.