Matematyka II sem. **Matematyka Dyskretna** **Lista 6a**

1. Pokazać, że
   (a) grafy G, H są krytyczne wtedy i tylko wtedy, gdy $\chi(G+H)$ jest krytyczny.
   (b) $\chi(G \Box H) = \max\{\chi(G), \chi(H)\}$.
   (c) $\chi(G \times H) \leqslant \min\{\chi(G), \chi(H)\}$.
2. Wyznaczyć indeks chromatyczny grafu K_n.

Poniższy tekst został zaczerpnięty z książki: A. Benjamin, G. Chartrand, P. Zhang, The Fascinating World of Graph Theory, Princeton University Press, 2015.

APPLICATIONS OF VERTEX COLORINGS

There are many problems that can be represented by a graph and whose solution involves finding the chromatic number of this graph. We present two examples in this section that should be reminiscent of examples that appeared in Chapter 1.

Example 11.8: *The mathematics department of a certain college plans to schedule the classes Graph Theory* (GT)*, Statistics* (S)*, Linear Algebra* (LA)*, Advanced Calculus* (AC)*, Geometry* (G) *and Modern Algebra* (MA) *this summer. Ten students have indicated the courses they plan to take.*

Anden: LA, S;	*Brynn:* MA, LA, G;
Chase: MA, G, LA;	*Denise:* G, LA, AC;
Everett: AC, LA, S;	*François:* G, AC;
Greg: GT, MA, LA;	*Harper:* LA, GT, S;
Irene: AC, S, LA;	*Jennie:* GT, S.

With this information, use graph theory to determine the minimum number of time periods needed to offer these courses so that every two classes having a student in common are taught at different time periods during the day. Of course, two classes having no students in common can be taught during the same period.

SOLUTION:

First, we construct a graph H whose vertices are the six subjects. Two vertices (subjects) are joined by an edge if some student is taking classes in these two subjects (see Figure 11.7). The minimum number of time periods is $\chi(H)$. Since H contains the odd cycle (GT, S, AC, G, MA, GT), it follows that three colors are needed to color the vertices on this cycle. Since LA is adjacent to all vertices of this cycle, a fourth color is needed for LA. Thus $\chi(H) \geq 4$. However, there is a 4-coloring of H shown in Figure 11.7 and so $\chi(H) = 4$. This also tells us one way to schedule these six classes during four time periods, namely period 1: Graph Theory, Advanced Calculus; period 2: Geometry; period 3: Statistics, Modern Algebra; period 4: Linear Algebra. ♦

Example 11.9: *Figure 11.8 shows the nine traffic lanes* L1, L2, ..., L9 *at the intersection of two busy streets. A traffic light is located at this intersection.*

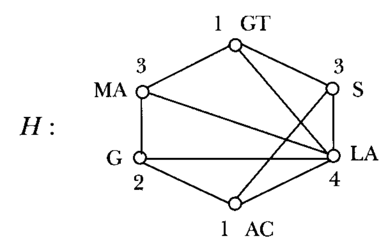

Figure 11.7. The graph of Example 11.8.

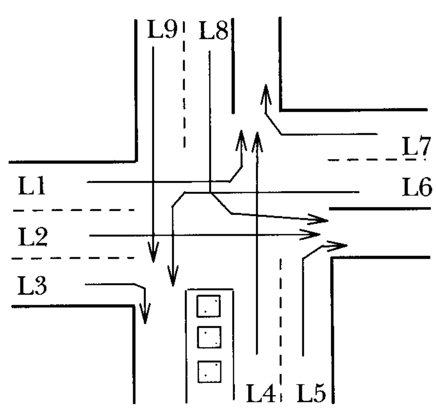

Figure 11.8. Traffic lanes at street intersections in Example 11.9.

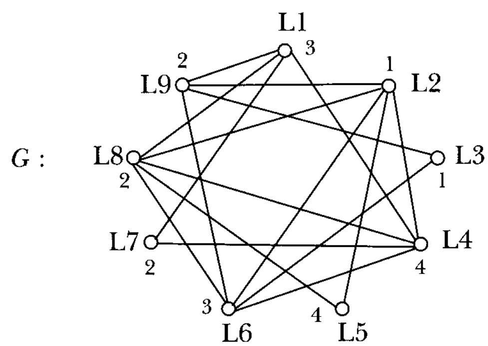

Figure 11.9. The graph in Example 11.9.

During a certain phase of the traffic light, those cars in lanes for which the light is green may proceed safely through the intersection. What is the minimum number of phases needed for the traffic light so that (eventually) all cars may proceed through the intersection?

SOLUTION:

First, a graph G is constructed that models this situation, where $V(G) = \{L1, L2, \ldots, L9\}$ and two vertices (lanes) are joined by an edge if vehicles in these two lanes cannot safely enter the intersection at the same time, as there is the possibility of an accident (see Figure 11.9).

Answering this question requires determining the chromatic number of the graph G in Figure 11.9. First, notice that the four vertices L2, L4, L6 and L8 are mutually adjacent and so four colors are needed to color these vertices. Thus $\chi(G) \geq 4$. Since there exists a proper coloring of G using the four colors 1, 2, 3, 4, as indicated in Figure 11.9, $\chi(G) = 4$.

Consequently, the minimum number of phases for the traffic light is four and vehicles in lanes with the same color may proceed through the intersection at the same time once the traffic light turns green for that phase. ♦

Example 12.5: *Five individuals have been invited to a bridge tournament (bridge is a game of cards): Allen (A), Brian (B), Charles (C), Doug (D) and Ed (E). A game of bridge is played between two two-person teams. Every two-person team {X, Y} is to play against all other two-person teams {W, Z}, where, of course, neither W nor Z is X or Y. If the same team cannot play bridge more than once on the same day, what is the fewest number of days needed for all possible games of bridge to be played. Set up a schedule for doing this in the smallest number of days. What graph models this situation?*

SOLUTION:

We construct a graph G whose vertices consist of all two-person teams, where we denote a vertex {X, Y} by XY for simplicity. Two vertices (two-person teams) XY and WZ are adjacent in G if they will be playing a game of bridge. The graph G is shown in Figure 12.3. Observe that G is the famous Petersen graph.

We have already seen that the Petersen graph G is not 1-factorable and so G is a class two graph. Thus $\chi'(G) = 4$. An edge coloring of G with four colors is shown in Figure 12.3. This creates the following schedule of games which takes place over the smallest number of days:

Day 1: AB–DE, AE–BC, AC–BE, AD–CE;
Day 2: AB–CE, AC–DE, AE–BD, AD–BC, BE–CD;
Day 3: AB–CD, BC–DE, BD–CE;
Day 4: AC–BD, AD–BE, AE–CD.

♦

G :

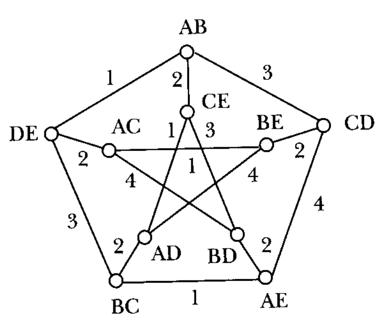

Figure 12.3. The graph in Example 12.5.