Lecture notes on LQR/LQG controller design

João P. Hespanha

February 27, 2005[1]

[1]Revisions from version January 26, 2005 version: Chapter 5 added.

Contents

Chapter 1

Linear Quadratic Regulation (LQR)

Summary

1. Problem definition
2. Solution to the LQR problem
3. LQR in Matlab

1.1 Deterministic Linear Quadratic Regulation (LQR)

Figure 1.1 shows the feedback configuration for the *Linear Quadratic Regulation (LQR) problem.*

Attention! Note the *negative feedback* and the *absence of a reference signal.*

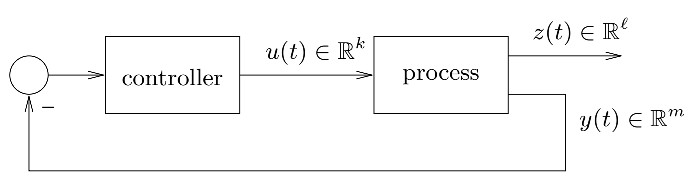


Figure 1.1. Linear quadratic regulation (LQR) feedback configuration

The process is assumed to be a continuous-time LTI system of the form:

$$\begin{aligned} \dot{x} &= Ax + Bu, & x &\in \mathbb{R}^n,\ u \in \mathbb{R}^k, \\ y &= Cx, & y &\in \mathbb{R}^m, \\ z &= Gx + Hu, & z &\in \mathbb{R}^\ell \end{aligned}$$

and has two distinct outputs:

1. The *measured output* $y(t)$ corresponds to the signal(s) that can be measured and are therefore available for control.
2. The *controlled output* $z(t)$ corresponds to a signal(s) that one would like to make as small as possible in the shortest possible amount of time.

Sometimes $z(t) = y(t)$ which means that our control objective is simply to make the measured output very small. Other times one may have

$$z(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix},$$

which means that we want to make both the measured output $y(t)$ and its derivative $\dot{y}(t)$ very small. Many other options are possible.

1.2 Optimal Regulation

The LQR problem is defined as follows: Find the control input $u(t)$, $t \in [0, \infty)$ that makes the following criterion as small as possible

$$J_{\text{LQR}} := \int_0^\infty \|z(t)\|^2 + \rho \, \|u(t)\|^2 dt, \tag{1.1}$$

where ρ is a positive constant. The term

$$\int_0^\infty \|z(t)\|^2 dt$$

corresponds to the *energy of the controlled output* and the term

$$\int_0^\infty \|u(t)\|^2 dt$$

to the *energy of the control signal.* In LQR one seeks a controller that minimizes both energies. However, decreasing the energy of the controlled output will require a large control signal and a small control signal will lead to large controlled outputs. The role of the constant ρ is to establish a trade-off between these conflicting goals:

1. When we chose ρ very large, the most effective way to decrease J_{LQR} is to use little control, at the expense of a large controlled output.
2. When we chose ρ very small, the most effective way to decrease J_{LQR} is to obtain a very small controlled output, even if this is achieved at the expense of a large controlled output.

Sidebar 1: A simple choice for the matrices Q and R is given by the *Bryson's rule...*

Often the optimal LQR problem is defined more generally and consists of finding the control input that minimizes

$$J_{\text{LQR}} := \int_0^\infty z(t)'\bar{Q}z(t) + \rho \, u'(t)\bar{R}u(t) \; dt, \tag{1.2}$$

where $Q \in \mathbb{R}^{\ell \times \ell}$ and $R \in \mathbb{R}^{m \times m}$ are symmetric positive-definite matrices and ρ a positive constant.

We shall consider the most general form for a quadratic criterion, which is

$$J := \int_0^\infty x(t)'Qx(t) + u'(t)Ru(t) + 2x'(t)Nu(t)dt. \tag{J-LQR}$$

Since $z = Gx + Hu$, the criterion in (1.1) is a special form of (J-LQR) with

$$Q = G'G, \qquad R = H'H + \rho I, \qquad N = G'H$$

and (1.2) is a special form of (J-LQR) with

$$Q = G'\bar{Q}G, \qquad R = H'\bar{Q}H + \rho\bar{R}, \qquad N = G'\bar{Q}H.$$

1.3 Solution to the LQR problem

Our present goal is to find a control input $u(t)$, $t \in [0, \infty)$ to

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k. \tag{AB-CLTI}$$

that minimizes the following quadratic criterion

$$J_{\text{LQR}} := \int_0^\infty x(t)'Qx(t) + u(t)'Ru(t) + 2x(t)'Nu(t)\, dt. \tag{J-LQR}$$

We will solve this problem using an argument based on "square completion" that avoids the use of calculus of variations. In essence, we will use purely algebraic manipulations to re-write the criterion (J-LQR) in the following form

$$J_{\text{LQR}} = J_0 + \int_0^\infty \big(u(t) - u_0(t)\big)' R \big(u(t) - u_0(t)\big)\, dt, \tag{1.3}$$

where the constant J_0 is a *feedback invariant* and $u_0(t)$ an appropriately selected signal. Since R is positive definite, we will conclude directly from (1.3) that (i) the minimum value for the criteria is equal to J_0 and (ii) the control $u(t) := u_0(t)$, $\forall t \geq 0$ achieves this minimum.

Notation 1: A quantity is called a *feedback invariant* if its value does not depend on the choice of the control input $u(t)$, $t \geq 0$.

1.3.1 Feedback invariants

It turns out that finding feedback invariants for the LTI system (AB-CLTI) is relatively straightforward:

Lemma 1 (Feedback invariant). *Let P be a symmetric matrix. For every control input $u(t)$, $t \in [0, \infty)$ for which $x(t) \to 0$ as $t \to \infty$, he have that*

$$\int_0^\infty x'(A'P + PA)x + 2x'PBu\, dt = -x(0)'Px(0). \tag{1.4}$$

Sidebar 2: This shows that the left-hand side of (1.4) is a feedback invariant for any u that drives x to zero.

Attention! To keep the formulas short, in the remainder of this section we drop the time dependence (t) when the state x and the input u appear inside time integrals.

Proof of Lemma 1. This result follows from the following simple computation:

$$\int_0^\infty x'(A'P + PA)x + 2x'PBu\, dt = \int_0^\infty (x'A' + u'B')Px + x'P(Ax + Bu)\, dt$$
$$= \int_0^\infty \dot{x}'Px + x'P\dot{x}\, dt = \int_0^\infty \frac{\mathrm{d}(x'Px)}{\mathrm{d}t} dt = \lim_{t\to\infty} x'(t)'Px(t) - x(0)'Px(0). \qquad \square$$

1.3.2 Square completion

To force the feedback invariant in Lemma 1 to appear in the expression for J_{LQR}, we add and subtract it from the right-hand side of (J-LQR):

$$J_{\text{LQR}} = x(0)'P\,x(0) + \int_0^\infty x'(A'P + PA + Q)x + u'Ru + 2x'(PB + N)u\, dt. \tag{1.5}$$

To re-write (1.5) as (1.3) we need to combine the all terms in u into a single quadratic form by square completion:

$$u'Ru + 2x'(PB + N)u = (u - u_0)'R(u - u_0) - x'(PB + N)R^{-1}(B'P + N')x$$

where

$$u_0 := -R^{-1}(B'P + N')x.$$

Replacing this in (1.5), we obtain

$$J_{\rm LQR} = J_0 + \int_0^\infty (u-u_0)' R (u-u_0)\; dt, \tag{1.6}$$

where

$$J_0 := x(0)' P\, x(0) + \int_0^\infty x' \big(A'P + PA + Q - (PB+N)R^{-1}(B'P+N')\big) x\; dt.$$

Notation 2: Equation (1.7) is called an *Algebraic Riccati Equation (ARE)*.

To make sure that J_0 is a feedback invariant, we ask that P be chosen so that

$$A'P + PA + Q - (PB+N)R^{-1}(B'P+N') = 0. \tag{1.7}$$

In this case, we conclude from (1.6) that the optimal control is given by

$$u(t) = u_0(t) = -R^{-1}(B'P+N')x(t), \quad \forall t \geq 0,$$

which results in the following closed-loop system

$$\dot{x} = \big(A - BR^{-1}(B'P+N')\big)x.$$

The following was proved:

Notation 3: Recall that a matrix is *Hurwitz* or a *stability matrix* if all its eigenvalues have negative real part.

Sidebar 3: Asymptotic stability of the closed-loop is needed because we assumed that $\lim_{t\to\infty} x(t)Px(t) = 0$.

Matlab hint 1: `lqr` solves the ARE (1.7) and computes the optimal state-feedback (1.8)...

Theorem 1. *Assume that there exists a symmetric solution P to the* Algebraic Riccati Equation (1.7) *for which $A - BR^{-1}(B'P+N')$ is Hurwitz. Then the feedback law*

$$u(t) := -Kx(t), \quad \forall t \geq 0, \qquad\qquad K := R^{-1}(B'P+N'), \tag{1.8}$$

minimizes the LQR criterion (J-LQR) *and leads to*

$$J_{\rm LQR} := \int_0^\infty x'Qx + u'Ru + 2x'Nu\; dt = x'(0)Px(0).$$

1.4 LQR in Matlab

Matlab Hint 1 (`lqr`). The command `[K,P,E]=lqr(A,B,Q,R,N)` solves the Algebraic Riccati Equation

$$\mathtt{A'P} + \mathtt{PA} + \mathtt{Q} - (\mathtt{PB}+\mathtt{N})R^{-1}(\mathtt{B'P}+\mathtt{N'}) = 0$$

and computes the (negative feedback) optimal state-feedback matrix gain

$$\mathtt{K} = \mathtt{R}^{-1}(\mathtt{B'P}+\mathtt{N'})$$

that minimizes the LQR criteria

$$J := \int_0^\infty x'\mathtt{Q}x + u'\mathtt{R}u + 2x'\mathtt{N}u\; dt.$$

for the continuous-time process

$$\dot{x} = \mathtt{A}x + \mathtt{B}u.$$

This command also returns the poles `E` of the closed-loop system

$$\dot{x} = (\mathtt{A} - \mathtt{BK})x. \qquad \square$$

1.5 Additional notes

Sidebar 1 (Bryson's rule). A reasonable simple choice for the matrices Q and R is given by the *Bryson's rule* [3, p. 537]: Select Q and R diagonal with

$$Q_{ii} = \frac{1}{\text{maximum acceptable value of } z_i^2}, \qquad i \in \{1, 2, \dots, \ell\},$$
$$R_{jj} = \frac{1}{\text{maximum acceptable value of } u_j^2}, \qquad j \in \{1, 2, \dots, k\},$$

which corresponds to the following criterion

$$J_{\text{LQR}} := \int_0^\infty \Big(\sum_{i=1}^{\ell} Q_{ii}\, z_i(t)^2 + \rho \sum_{j=1}^{m} R_{jj}\, u(t)^2 \Big) dt.$$

In essence, the Bryson's rule scales the variables that appear in J_{LQR} so that the *maximum acceptable value for each term is one*. This is especially important when the units used for the different components of u and z make the values for these variables numerically very different from each other.

Although Bryson's rule usually gives good results, often it is just the starting point to a trial-and-error iterative design procedure aimed at obtaining desirable properties for the closed-loop system. We shall pursue this in Section 3.3. □

Chapter 2

Algebraic Riccati Equation (ARE)

Summary

1. Hamiltonian matrix
2. Domain of the Riccati operator
3. Stable subspace of the Hamiltonian matrix

2.1 Hamiltonian matrix

The construction of the optimal LQR feedback law in Theorem 1 required the existence of a symmetric solution P to the ARE

$$A'P + PA + Q - (PB + N)R^{-1}(B'P + N') = 0 \tag{2.1}$$

for which $A - BR^{-1}(B'P + N')$ is Hurwitz. To study the solutions of this equation it is convenient to expand the last term in the left-hand-side of (2.1), which leads to

$$(A - BR^{-1}N')'P + P(A - BR^{-1}N') + Q - NR^{-1}N' - PBR^{-1}B'P = 0. \tag{2.2}$$

This equation can be re-written compactly as

$$\begin{bmatrix} P & -I \end{bmatrix} \mathbf{H} \begin{bmatrix} I \\ P \end{bmatrix} = 0. \tag{2.3}$$

where

$$\mathbf{H} := \begin{bmatrix} A - BR^{-1}N' & -BR^{-1}B' \\ -Q + NR^{-1}N' & -(A - BR^{-1}N')' \end{bmatrix} \in \mathbb{R}^{2n \times 2n}.$$

is called the *Hamiltonian matrix associated with* (2.1).

2.2 Domain of the Riccati operator

A Hamiltonian matrix $\mathbf{H}$ is said to be in the *domain of the Riccati operator* if there exist symmetric matrices $\mathbf{H}_-, P \in \mathbb{R}^{n \times n}$ such that

Notation 4: We write $\mathbf{H} \in \text{Ric}$ when $\mathbf{H}$ is in the domain of the Riccati operator.

$$\mathbf{H}\begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} \mathbf{H}_-, \tag{2.4}$$

with $\mathbf{H}_-$ Hurwitz and I the $n \times n$ identity matrix.

Theorem 2. *Suppose that* $\mathbf{H}$ *is in the domain of the Riccati operator and let* $P, \mathbf{H}_- \in \mathbb{R}^{n\times n}$ *be as in* (2.4).

Notation 5: In general the ARE has multiple solutions, but only the one in (2.4) makes the closed-loop asymptotically stable. This solution is called the *stabilizing solution.*

(i) P *satisfies* (2.1),

(ii) $A - BR^{-1}(B'P + N') = \mathbf{H}_-$ *is Hurwitz, and*

(iii) P *is symmetric matrix.* □

Proof of Theorem 2. To prove (i), we left-multiply (2.4) by $\begin{bmatrix} P & -I \end{bmatrix}$ and obtain (2.3).

To prove (ii), we just look at the top n rows of the matrix equation (2.4):

$$\begin{bmatrix} A - BR^{-1}N' & -BR^{-1}B' \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ \cdot \end{bmatrix} \mathbf{H}_-,$$

from which $A - BR^{-1}(B'P + N') = \mathbf{H}_-$ follows.

To prove (iii), we left-multiply (2.4) by $\begin{bmatrix} -P' & I \end{bmatrix}$ and obtain

$$\begin{bmatrix} -P' & I \end{bmatrix} \mathbf{H} \begin{bmatrix} I \\ P \end{bmatrix} = (P - P')\mathbf{H}_-.$$

Exercise 1: Check! $\left[\begin{smallmatrix} -P' & I \end{smallmatrix}\right] \mathbf{H} \left[\begin{smallmatrix} I \\ P \end{smallmatrix}\right] = -P'(A - BR^{-1}N')P + P'BR^{-1}B'P - Q + NR^{-1}N' - (A - BR^{-1}N')'P.$

Moreover, using the definition of $\mathbf{H}$ we also conclude that the left-hand side of this equation is symmetric and therefore

$$(P - P')\mathbf{H}_- = \mathbf{H}_-'(P' - P) = -\mathbf{H}_-'(P - P'). \tag{2.5}$$

Using this property repeatedly we further conclude that

$$(P - P')\mathbf{H}_-^k = -\mathbf{H}_-'(P' - P)\mathbf{H}_-^{k-1} = \cdots = (-\mathbf{H}_-')^k(P' - P), \quad k \in \{0, 1, 2, \dots\}. \tag{2.6}$$

To proceed, let

$$\Delta(s) = (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n), \quad \Re[\lambda_k] < 0.$$

denote the characteristic polynomial of $\mathbf{H}_-$ and the λ_k its eigenvalues. Using (2.6) and the Cayley-Hamilton Theorem, we conclude that

$$(P - P')\underbrace{\Delta(\mathbf{H}_-)}_{=0} = \Delta(-\mathbf{H}_-')(P' - P) = 0.$$

The result follows because $\Delta(-\mathbf{H}_-')$ is nonsingular and therefore $P' - P = 0$. To verify that $\Delta(-\mathbf{H}_-')$ is indeed nonsingular, note that

$$\begin{aligned} \Delta(-\mathbf{H}_-') &= (-\mathbf{H}_-' - \lambda_1 I)(-\mathbf{H}_-' - \lambda_2 I)\cdots(-\mathbf{H}_-' - \lambda_n I) \\ &= (-1)^n(\mathbf{H}_-' + \lambda_1 I)(\mathbf{H}_-' + \lambda_2 I)\cdots(\mathbf{H}_-' + \lambda_n I) \end{aligned}$$

and, since all the $-\lambda_k$ must have positive real, none can be an eigenvalue of $\mathbf{H}_-'$ and therefore all the matrices $(-\mathbf{H}_-' - \lambda_k I)$ must be nonsingular. ■

2.3 Stable subspaces

Given a square matrix M, suppose that we factor its characteristic polynomial as the following product of polynomials

$$\Delta(s) = \det(sI - M) = \Delta_-(s)\Delta_+(s),$$

where all the roots of $\Delta_-(s)$ have negative real part and all roots of $\Delta_+(s)$ have positive or zero real parts. The *stable subspace* of M is defined by

$$\mathcal{V}_- := \operatorname{Ker} \Delta_-(M).$$

Exercise 2: Show that $\mathcal{V}_-$ is an M-invariant subspace. Since $\Delta_-(M)M = M\Delta_-(M)$, we have that $x \in \mathcal{V}_- \Rightarrow \Delta_-(M)x = 0 \Rightarrow M\Delta_-(M)x = 0 \Rightarrow \Delta_-(M)Mx = 0 \Rightarrow Mx \in \mathcal{V}_-$.

Properties (Stable subspaces). Let $\mathcal{V}_-$ be the stable subspace of M

P1 $\dim \mathcal{V}_- = \deg \Delta_-(s)$.

P2 For every matrix V whose columns form a basis for $\mathcal{V}_-$, there exists a Hurwitz matrix M_- whose characteristic polynomial is $\Delta_-(s)$ such that

$$MV = VM_-. \tag{2.7}$$

Sidebar 4: Therefore the dimension of $\mathcal{V}_-$ is the number of eigenvalues of M with negative real-part (with repetitions).

Exercise 3 (Stable subspaces). Prove Properties P1–P2.

Hint: Transform M into its Jordan normal form □

Solution to Exercise 3. To prove P1, consider the Jordan normal form of M

$$M = T \begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix} T^{-1},$$

where T is a nonsingular matrix, J_- contains all Jordan blocks corresponding to eigenvalues with negative real part, and J_+ the remaining blocks. The eigenvalues of J_- are precisely the roots of $\Delta_-(s)$ and therefore (i) $\Delta_-(s)$ is the characteristic polynomial of J_- and (ii) the size of J_- is equal to $\deg \Delta_-(s)$. Since

$$M^k = T \begin{bmatrix} J_-^k & 0 \\ 0 & J_+^k \end{bmatrix} T^{-1},$$

we conclude that

$$\Delta_-(M) = T \begin{bmatrix} \Delta_-(J_-) & 0 \\ 0 & \Delta_-(J_+) \end{bmatrix} T^{-1} = T \begin{bmatrix} 0 & 0 \\ 0 & \Delta_-(J_+) \end{bmatrix} T^{-1},$$

where the second equality is a consequence of the Cayley-Hamilton theorem. Therefore

$$x \in \operatorname{Ker} \Delta_-(M) \quad \Leftrightarrow \quad T^{-1}x \in \operatorname{Ker} \begin{bmatrix} 0 & 0 \\ 0 & \Delta_-(J_+) \end{bmatrix}$$

Since no root of $\Delta_-(s)$ is an eigenvalue of J_+, the matrix $\Delta_-(J_+)$ must be nonsingular and

$$\operatorname{Ker} \begin{bmatrix} 0 & 0 \\ 0 & \Delta_-(J_+) \end{bmatrix} = \operatorname{Im} \begin{bmatrix} I \\ 0 \end{bmatrix},$$

where I denotes an identity matrix with the size of J_-, i.e., the degree of $\Delta_s(s)$. We thus conclude that

$$x \in \operatorname{Ker} \Delta_-(M) \quad \Leftrightarrow \quad T^{-1}x \in \operatorname{Im} \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad x \in \operatorname{Im} T \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Exercise 4: Why? Factoring $\Delta_-(s) = (\lambda_1 - s) \cdots (\lambda_r - s)$, we have $\Delta_-(J_+) = (\lambda_1 I - J_+) \cdots (\lambda_r I - J_+)$ and all the matrices in this product are nonsingular because none of the λ_i is an eigenvalue of J_+.

Sidebar 5: Because the top component of a vector in the kernel can take any value but the lower component must be zero.

Since T is nonsingular, the columns of $T\begin{bmatrix} I & 0 \end{bmatrix}'$ are linearly dependent and therefore form a basis for T. Since the number of columns of this matrix is equal to the degree of $\Delta_-(s)$, we conclude that $\dim \mathcal{V}_- = \deg \Delta_-(s)$.

To prove P2, note that we just concluded that $T\begin{bmatrix} I & 0 \end{bmatrix}'$ is a basis for $\mathcal{V}_-$ and that

$$MT\begin{bmatrix} I \\ 0 \end{bmatrix} = T\begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix} T^{-1}T\begin{bmatrix} I \\ 0 \end{bmatrix} = T\begin{bmatrix} I \\ 0 \end{bmatrix} J_-. \tag{2.8}$$

Since the columns of V form another basis for $\mathcal{V}_-$, these two basis must be related by a nonsingular coordinate transformation matrix P and therefore

$$VP = T\begin{bmatrix} I \\ 0 \end{bmatrix},.$$

From this and (2.8) we obtain

$$MVP = VPJ_- \quad \Leftrightarrow \quad MV = VPJ_P^{-1},$$

and therefore (2.7) holds for $M_- := PJ_P^{-1}$. Since M_- and J_- are related by a similarity transformation, they have the same characteristic polynomial and the same eigenvalues. ■

2.4 Stable subspace of the Hamiltonian matrix

Our goal is now to find conditions under which the Hamiltonian matrix $\mathbf{H} \in \mathbb{R}^{2n\times 2n}$ belongs to the domain of the Riccati operator, i.e., for which there exist symmetric matrices $\mathbf{H}_-, P \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{H}\begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix}\mathbf{H}_-,$$

with $\mathbf{H}_-$ Hurwitz and I the $n \times n$ identity matrix. From the properties of stable subspaces, we conclude that such a matrix $\mathbf{H}_-$ exists if (i) the dimension of the stable subspace $\mathcal{V}_-$ of $\mathbf{H}$ is equal to n and (ii) we can find a basis for this space of the form $\begin{bmatrix} I & P' \end{bmatrix}'$.

2.4.1 Dimension of the stable subspace of H.

To investigate the dimension of $\mathcal{V}_-$ we need to compute its characteristic polynomial $\Delta(s)$. To this effect, note that

$$\mathbf{H}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \begin{bmatrix} BR^{-1}B' & A - BR^{-1}N' \\ (A - BR^{-1}N')' & -Q + NR^{-1}N' \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}\mathbf{H}'.$$

Therefore, defining $J := \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$,

$$\mathbf{H} = -J\mathbf{H}'J^{-1}.$$

Since the characteristic polynomial in invariant with respect to similarity transformations and matrix transposition, we conclude that

$$\begin{aligned} \Delta(s) := \det(sI - \mathbf{H}) = \det(sI + J\mathbf{H}'J^{-1}) &= \det(sI + \mathbf{H}') \\ &= \det(sI + \mathbf{H}) = (-1)^{2n}\det((-s)I - \mathbf{H}) = \Delta(-s), \end{aligned}$$

which shows that if λ is an eigenvalue of $\mathbf{H}$ then $-\lambda$ is also an eigenvalue of $\mathbf{H}$ with the same multiplicity. We thus conclude that the $2n$ eigenvalues of $\mathbf{H}$ are distributed symmetrically with respect to the imaginary axis. To check that we actually have n eigenvalues with negative real part and another n with positive real part, we need to make sure that $\mathbf{H}$ has no eigenvalues over the imaginary axis:

Lemma 2. *Assume that $Q - NR^{-1}N' \geq 0$. When the pair (A, B) is stabilizable and the pair $(A - BR^{-1}N', Q - NR^{-1}N')$ is detectable we have*

(i) the Hamiltonian matrix $\mathbf{H}$ has no eigenvalues on the imaginary axis

(ii) its stable subspace $\mathcal{V}_-$ has dimension n. □

Attention! The best LQR controllers are obtained for choices of the controlled output z for which $N = G'H = 0$ (cf. Chapter 3). In this case, the sole conditions simplify to stabilizability of (A, B) and detectability of (A, G). □

Exercise 5: Show that detectability of (A, G) is equivalent to detectability of (A, Q) with $Q := G'G$. *Hint: use the eigenvector test and note that the kernels of G and $G'G$ are exactly the same.*

Proof of Lemma 2. By contradiction, let $x := \begin{bmatrix} x_1' & x_2' \end{bmatrix}'$, $x_1, x_2 \in \mathbb{C}^n$ be an eigenvector of $\mathbf{H}$ associated with an eigenvalue $\lambda := j\omega$, $\omega \in \mathbb{R}$. This means that

$$\begin{bmatrix} j\omega I - A + BR^{-1}N' & BR^{-1}B' \\ Q - NR^{-1}N' & j\omega + (A - BR^{-1}N')' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \tag{2.9}$$

Using the fact that (λ, x) is an eigenvalue/eigenvector pair of $\mathbf{H}$, one concludes that

Notation 6: $(\cdot)^*$ denotes transpose complex conjugate.

Attention! The notation used here differs from that of MATLAB. Here $(\cdot)'$ denotes transpose and $(\cdot)^*$ transpose complex conjugate, whereas in MATLAB, $(\cdot).'$ denotes transpose and $(\cdot)'$ transpose complex conjugate.

$$\begin{aligned} \begin{bmatrix} x_2^* & x_1^* \end{bmatrix} \mathbf{H} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1^* & x_2^* \end{bmatrix} \mathbf{H}' \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} &= \begin{bmatrix} x_2^* & x_1^* \end{bmatrix} (\mathbf{H}x) + (\mathbf{H}x)^* \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \\ &= \begin{bmatrix} x_2^* & x_1^* \end{bmatrix} j\omega \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(j\omega \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)^* \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \\ &= j\omega(x_2^* x_1 + x_1^* x_2) - j\omega(x_1^* x_2 + x_2^* x_1) = 0. \end{aligned} \tag{2.10}$$

On the other hand, using the definition of $\mathbf{H}$ one concludes that the left-hand side of (2.10) is given by

$$\begin{aligned} &\begin{bmatrix} x_2^* & x_1^* \end{bmatrix} \begin{bmatrix} A - BR^{-1}N' & -BR^{-1}B' \\ -Q + NR^{-1}N' & -(A - BR^{-1}N')' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} x_1^* & x_2^* \end{bmatrix} \begin{bmatrix} (A - BR^{-1}N')' & -Q + NR^{-1}N' \\ -BR^{-1}B' & -(A - BR^{-1}N') \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \\ &\quad = -2x_1^*(Q - NR^{-1}N')x_1 - 2x_2^*(BR^{-1}B')x_2. \end{aligned}$$

Since this expression must equal zero and $R^{-1} > 0$, we conclude that

$$(Q - NR^{-1}N')x_1 = 0, \qquad B'x_2 = 0.$$

From this and (2.9) we also conclude that

$$(j\omega I - A + BR^{-1}N')x_1 = 0, \qquad (j\omega + A')x_2 = 0.$$

But then we found an eigenvector x_2 of A' in the kernel of B' and an eigenvector x_1 of $A - BR^{-1}N'$ in the kernel of $Q - NR^{-1}N'$. Since the corresponding eigenvalues do not have negative real part, this contradicts the stabilizability and detectability assumptions.

The fact that $\mathcal{V}_-$ has dimension n follows from the discussion preceding the statement of the lemma. ■

2.4.2 Basis for the stable subspace of H.

Suppose that the assumptions of Lemma 2 hold and let

$$V := \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \in \mathbb{R}^{2n\times n}$$

be a matrix whose n columns form a basis for the stable subspace $\mathcal{V}_-$ of $\mathbf{H}$. Assuming that $V_1 \in \mathbb{R}^{n\times n}$ is nonsingular, then

Sidebar 6: Under the assumptions of Lemma 2, this is always true as shown in [2, Theorem 6.5, p. 202].

$$VV_1^{-1} = \begin{bmatrix} I \\ P \end{bmatrix}, \qquad P := V_2V_1^{-1}$$

is also a basis for $\mathcal{V}_-$. Therefore, we conclude from property P2 that there exists a Hurwitz matrix $\mathbf{H}_-$ such that

$$\mathbf{H}\begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix}\mathbf{H}_-, \tag{2.11}$$

and therefore $\mathbf{H}$ belongs to the domain of the Riccati operator. Combining Lemma 2 with Theorem 2 we obtain the following main result regarding the solution to the ARE:

Theorem 3. *Assume that $Q - NR^{-1}N' \geq 0$. When the pair (A, B) is stabilizable and the pair $(A - BR^{-1}N', Q - NR^{-1}N')$ is detectable we have*

(i) $\mathbf{H}$ *is in the domain of the Riccati operator,*

(ii) P *is symmetric*

(iii) P *satisfies* (2.1),

(iv) $A - BR^{-1}(B'P + N') = \mathbf{H}_-$ *is Hurwitz,*

Sidebar 7: When the pair $(A-BR^{-1}N', Q-NR^{-1}N')$ is observable, one can show that P is also positive definite...

where $P, \mathbf{H}_- \in \mathbb{R}^{n\times n}$ are as in (2.11)*. Moreover, the eigenvalues of $\mathbf{H}_-$ are the eigenvalues of $\mathbf{H}$ with negative real part.* □

Attention! Going back to the minimization of

$$J_{\mathrm{LQR}} := \int_0^\infty z'\bar{Q}z + \rho\, u'\bar{R}u\, dt, \quad z := Gx + Hu \quad \rho, Q, R > 0,$$

which corresponds to

$$Q = G'\bar{Q}G, \qquad R = H'\bar{Q}H + \rho\bar{R}, \qquad N = G'\bar{Q}H.$$

When $N = 0$, we conclude that Theorem 3 requires the detectability of the pair $(A, Q) = (A, G'\bar{Q}G)$. Since $\bar{Q} > 0$ it is straightforward to verify (e.g., using the eigenvector test) that this is equivalent to the detectability of the pair (A, G), which means that the system must be detectable through the controlled output z.

The need for (A, B) to be stabilizable is quite reasonable because otherwise it is not possible to make $x \to 0$ for every initial condition. The need for (A, G) to be detectable can be intuitively understood by the fact that if the system had unstable modes that did not appear in z, it could be possible to make J_{LQR} very small, even though the state x would be exploding. □

Sidebar 7. To prove that P is positive definite, we re-write the ARE (2.2)

$$(A - BR^{-1}N')'P + P(A - BR^{-1}N') + Q - NR^{-1}N' - PBR^{-1}B'P = 0$$

as

$$\mathbf{H}'_{-}P + P\mathbf{H}_{-} = -S, \quad S := (Q - NR^{-1}N') + PBR^{-1}B'P.$$

The positive definiteness of P then follows from the observability Lyapunov test as long as we are able to establish the observability of the pair (H_{-}, S).

To show that the pair (H_{-}, S) is indeed observable we use the eigenvector test. By contradiction assume that x is an eigenvector of H_{-} that lies in the kernel of S, i.e.,

$$\big(A - BR^{-1}(B'P + N')\big)x = \lambda x, \qquad \big((Q - NR^{-1}N') + PBR^{-1}B'P\big)x = 0.$$

These equations imply that $(Q - NR^{-1}N')x = 0$ and $B'Px = 0$ and therefore

$$(A - BR^{-1}N')x = \lambda x, \qquad \big((Q - NR^{-1}N')x = 0,$$

which contradicts the fact that the pair $(A - BR^{-1}N', Q - NR^{-1}N')$ is observable. $\square$

Chapter 3

Frequency-domain and asymptotic properties of LQR

Summary

1. Kalman's inequality: complementary sensitivity function, Nyquist plot (SISO), gain and phase margins (SISO)
2. Loop-shaping using LQR
3. Cheap control asymptotic case: closed-loop poles and cost
4. LQR design example

3.1 Kalman's equality

Consider the following continuous-time LTI process

$$\dot{x} = Ax + Bu, \qquad z = Gx + Hu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ z \in \mathbb{R}^\ell, \tag{AB-CLTI}$$

for which one wants to minimize the following LQR criterion

$$J_{\text{LQR}} := \int_0^\infty \|z(t)\|^2 + \rho \|u(t)\|^2 \, dt, \tag{3.1}$$

where ρ is positive constant. Throughout this whole chapter we shall assume that

$$N := G'H = 0, \tag{3.2}$$

Sidebar 8: This condition is not being added for simplicity, without it the results in this section may not be valid.

for which the optimal control is given by

$$u = -Kx, \qquad K := R^{-1}B'P, \qquad R := H'H + \rho I,$$

where P is the stabilizing solution to the following ARE

$$A'P + PA + G'G - PBR^{-1}B'P = 0.$$

We saw in the Chapter 1 that under appropriate stabilizability and detectability assumptions, the LQR control results in a closed-loop system that is asymptotically stable.

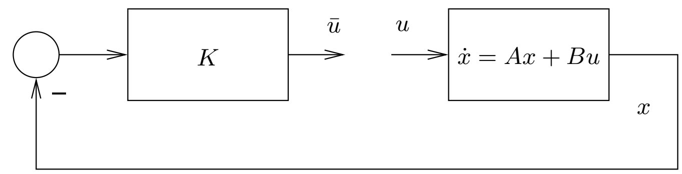


Figure 3.1. State-feedback open-loop gain

LQR controllers also have desirable properties in the frequency domain. To understand why, consider the open-loop transfer-matrix from the process' input u to the controller's output $\bar{u}$ (Figure 3.1). The state-space model from u to $\bar{u}$ is given by

$$\dot{x} = Ax + Bu, \qquad \bar{u} = -Kx,$$

which corresponds to the following open-loop *negative* feedback $k \times k$ transfer-matrix

$$\hat{L}(s) = K(sI - A)^{-1}B.$$

Another important open-loop transfer function is that from the control signal u to the controlled output z:

$$\hat{G}(s) = G(sI - A)^{-1}B + H.$$

These transfer functions are related by the so-called *Kalman's equality*:

Sidebar 9: Kalman's equality follows directly from simple algebraic manipulations of the ARE (cf. Exercise 6).

Kalman's equality. *For the LQR criterion in* (3.1) *with* (3.2)*, we have that*

$$\big(I + \hat{L}(-s)'\big)R\big(I + \hat{L}(s)\big) = R + H'H + \hat{G}(-s)'\hat{G}(s). \tag{3.3}$$

Kalman's equality has many important consequences. One of them is *Kalman's inequality*, which is obtained by setting $s = j\omega$ in (3.3) and using the fact that for real-rational transfer functions

$$\hat{L}(-j\omega)' = \hat{L}(j\omega)^*, \qquad \hat{G}(-j\omega)' = \hat{G}(j\omega)^*, \qquad H'H + \hat{G}(j\omega)^*\hat{G}(j\omega) \geq 0.$$

Kalman's inequality. *For the LQR criterion in* (3.1) *with* (3.2)*, we have that*

$$\big(I + \hat{L}(j\omega)\big)^* R\big(I + \hat{L}(j\omega)\big) \geq R, \quad \forall \omega \in \mathbb{R}. \tag{3.4}$$

Exercise 6 (Kalman equality). Prove Kalman's equality (3.3).

Hint: Add and subtract sP to the ARE and then left and right-multiply it by $-B'(sI+A')^{-1}$ and $(sI - A)^{-1}B$, respectively. □

Solution to Exercise 6. Following the hint, we first conclude that

$$(sI + A')P - P(sI - A) + G'G - PBR^{-1}B'P = 0.$$

and then

$$- B'P(sI - A)^{-1}B + B'(sI + A')^{-1}PB - B'(sI + A')^{-1}G'G(sI - A)^{-1}B \\ + B'(sI + A')^{-1}PBR^{-1}B'P(sI - A)^{-1}B = 0,$$

which can be re-written as

$$-R\hat{L}(s) - \hat{L}(-s)'R + (\hat{G}(-s)' - H')(\hat{G}(s) - H) - \hat{L}(-s)'R\hat{L}(s) = 0.$$

Using the facts that $H'G = 0$ and $G'H = 0$, we conclude that $H'(\hat{G}(s) - H) = 0$ and $(\hat{G}(-s)' - H')H = 0$. Therefore, we can further simply the equation above to

$$R\hat{L}(s) + \hat{L}(-s)'R + \hat{L}(-s)'R\hat{L}(s) = H'H + \hat{G}(-s)'\hat{G}(s),$$

which is equivalent to (3.4). ■

3.2 Frequency-domain properties: Single-input case

We focus our attention in single-input processes ($k = 1$), for which $\hat{L}(s)$ is a scalar transfer function. Dividing both sides of Kalman's inequality (3.4) by the scalar R, we obtain:

$$|1 + \hat{L}(j\omega)| \geq 1, \quad \forall \omega \in \mathbb{R},$$

which expresses the fact that *the Nyquist plot of* $\hat{L}(j\omega)$ *does not enter a circle of radius one around* -1. This is represented graphically in Figure 3.2 and has several significant implications, which are discussed next.

Sidebar 10: For multiple input systems similar conclusions could be drawn based on the *Multi-variable Nyquist Criterion...*

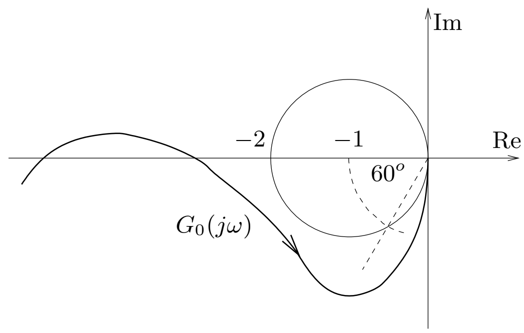


Figure 3.2. Nyquist plot for a LQR state-feedback controller

Positive gain margin If the process' gain is multiplied by a constant $k > 1$, its Nyquist plot simply expands radially and therefore the number of encirclements does not change. This corresponds to a *positive gain margin of* $+\infty$.

Negative gain margin If the process' gain is multiplied by a constant $.5 < k < 1$, its Nyquist plot contracts radially but the number of encirclements still does not change. This corresponds to a *negative gain margin of* $20\log_{10}(.5) = -6\,dB$.

Phase margin If the process' phase increases by $\theta \in [-60, 60]$ degrees, its Nyquist plots rotates by θ but the number of encirclements still does not change. This corresponds to a *phase margin of* ± 60 *degrees.*

Sensitivity and complementary sensitivity functions The sensitivity and the complementary sensitivity functions are given by

$$\hat{S}(s) = \frac{1}{1 + \hat{L}(s)}, \qquad \hat{T}(s) = \frac{\hat{L}(s)}{1 + \hat{L}(s)},$$

respectively. Kalman's inequality guarantees that

$$|\hat{S}(j\omega)| \leq 1, \quad |\hat{T}(j\omega) - 1| \leq 1, \quad |\hat{T}(j\omega)| \leq 2, \quad \Re[\hat{T}(j\omega)] \geq 0, \quad \forall \omega \in \mathbb{R}. \tag{3.5}$$

Sidebar 11: The first inequality results directly from the fact that $|1 + \hat{L}(j\omega)| \geq 1$, the second from the fact that $\hat{T}(s) = 1 - \hat{S}(s)$, and the last two from the fact that the second inequality shows that $\hat{T}(j\omega)$ must belong to a circle of radius one around $+1$.

We recall that

1. A small sensitivity function is desirable for good disturbance rejection. Generally, this is especially important at *low frequencies.*
2. A complementary sensitivity function close to one is desirable for good reference tracking. Generally, this is especially important at *low frequencies.*

3. A small complementary sensitivity function is desirable for good noise rejection. Generally, this is especially important at *high frequencies*.

Attention! Kalman's inequality is only valid when $N = 0$. When this is not the case, LQR controllers can exhibit significantly worst properties in terms of gain and phase margins. To same extent this limits the controlled outputs that should be placed in z. □

3.3 Loop-shaping using LQR: Single-input case

Using Kalman's inequality, we saw that any LQR controller automatically provides some upper bounds on the magnitude of the sensitivity function and its complementary. However, these bounds are frequency independent and may not result in appropriate loop-shaping.

We will see next a few rules that allow us to actually do loop-shaping using LQR. We continue to restrict our attention to the single-input case ($k = 1$).

Matlab hint 2: `sigma(sys)` draws the norm-Bode plot of the system `sys`...

Low-frequency open-loop gain Dividing both sides of Kalman's equality (3.3) by the scalar $R := H'H + \rho$, we obtain

$$|1 + \hat{L}(j\omega)|^2 = 1 + \frac{H'H}{H'H + \rho} + \frac{\|\hat{G}(j\omega)\|^2}{H'H + \rho}.$$

Therefore, for the range of frequencies for which $|\hat{L}(j\omega)| \gg 1$ (typically low frequencies),

$$|\hat{L}(j\omega)| \approx |1 + \hat{L}(j\omega)| \approx \frac{\|\hat{G}(j\omega)\|}{\sqrt{H'H + \rho}},$$

which means that the open-loop gain for the optimal feedback $\hat{L}(s)$ follows the "shape" of the Bode plot from u to the controlled output z.

This shows that one can shape the open-loop gain at low frequencies by selecting an appropriate controlled output z. For example, selecting $z := \begin{bmatrix} y & \gamma\dot{y} \end{bmatrix}'$ with $y := Cx$ scalar, i.e.,

$$z = \begin{bmatrix} y \\ \gamma\dot{y} \end{bmatrix} = \begin{bmatrix} Cx \\ \gamma CAx + \gamma CBu \end{bmatrix} \quad \Rightarrow \quad G = \begin{bmatrix} C \\ \gamma CA \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ \gamma CB \end{bmatrix}$$

we conclude that

$$\hat{G}(s) = \begin{bmatrix} \hat{P}(s) \\ \gamma s\hat{P}(s) \end{bmatrix} = \begin{bmatrix} 1 \\ \gamma s \end{bmatrix} \hat{P}(s), \qquad \hat{P}(s) := C(sI - A)^{-1}B,$$

and therefore

$$|\hat{L}(j\omega)| \approx \frac{\sqrt{1 + \gamma^2\omega^2}\,|\hat{P}(j\omega)|}{\sqrt{H'H + \rho}} = \frac{|1 + j\gamma\omega|\,|\hat{P}(j\omega)|}{\sqrt{H'H + \rho}}.$$

In this case, the low-frequency open-loop gain mimics the process transfer function from u to y, with an extra zero at $1/\gamma$ and scaled by $\frac{1}{\sqrt{H'H+\rho}}$. Thus

1. ρ moves the magnitude Bode plot up and down (more precisely $H'H + \rho$),
2. large values for γ lead to a low-frequency zero and generally result in a larger phase margin (above the minimum of 60 degrees) and smaller overshoot in the step response. However, this is often achieved at the expense of a slower response.

High-frequency open-loop gain Figure 3.2 shows that the open-loop gain $\hat{L}(j\omega)$ can have at most -90^o phase for high-frequencies and therefore the "roll-off" rate is at most -20dB/decade. In practice, this means that for $\omega \gg 1$, we have

$$|\hat{L}(j\omega)| \approx \frac{c}{\omega\sqrt{H'H+\rho}},$$

for some constant c and therefore the cross-over frequency is approximately given by

$$\frac{c}{\omega_{\text{cross}}\sqrt{H'H+\rho}} \approx 1 \quad \Leftrightarrow \quad \omega_{\text{cross}} \approx \frac{c}{\sqrt{H'H+\rho}}.$$

Thus

1. LQR controllers always exhibit a high-frequency magnitude decay of -20dB/decade,
2. the cross-over frequency is proportional to $1/\sqrt{H'H+\rho}$ and generally small values for $H'H+\rho$ result in faster step responses.

Attention! The (slow) -20dB/decade magnitude decrease is the main shortcoming of state-feedback LQR controllers because it may not be sufficient to clear high-frequency upper bounds on the open-loop gain needed to reject disturbances and/or for robustness with respect to process uncertainty. We will see in Section 4.5 that this can actually be improved with output-feedback controllers. □

3.4 Cheap control asymptotic case

In view of the LQR criterion

$$J_{\text{LQR}} := \int_0^\infty \|z(t)\|^2 + \rho\|u(t)\|^2 \, dt,$$

by making ρ very small one does not penalize the energy used by the control signal. Based on this, one could expect that as $\rho \to 0$

(i) the system's response becomes arbitrarily fast, and

(ii) the optimal value of the criterion converges to zero.

This limiting case is called *cheap control* and it turns out that whether or not the above conjectures are true depends on the transmission zeros of the system.

3.4.1 Closed-loop poles

We saw in Chapter 2 that the poles of the closed-loop correspond to the stable eigenvalues of the Hamiltonian matrix

$$\mathbf{H} := \begin{bmatrix} A & -BR^{-1}B' \\ -G'G & -A' \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \qquad R := H'H + \rho I \in \mathbb{R}^{k\times k}$$

(cf. Theorem 3). To determine the eigenvalues of $\mathbf{H}$, we use the fact that

Sidebar 12: Cf. Exercise 7.

$$\det(sI - \mathbf{H}) = c\Delta(s)\Delta(-s)\det\Big(R - H'H + \hat{G}(-s)'\hat{G}(s)\Big). \tag{3.6}$$

where $c := (-1)^n \det R^{-1}$ and

$$\Delta(s) := \det(sI - A), \qquad \hat{G}(s) := G(sI-A)^{-1}B + H.$$

As $\rho \to 0$, $H'H \to R$ and therefore

$$\det(sI - \mathbf{H}) \to c\Delta(s)\Delta(-s)\det \hat{G}(-s)'\hat{G}(s). \tag{3.7}$$

We saw that there exist unimodular real polynomial matrices $L(s) \in \mathbb{R}[s]^{\ell\times\ell}$, $R(s) \in \mathbb{R}[s]^{k\times k}$ such that

$$\hat{G}(s) = L(s)\,\mathrm{SM}_G(s)R(s), \tag{3.8}$$

where

$$\mathrm{SM}_G(s) := \begin{bmatrix} \frac{\eta_1(s)}{\psi_1(s)} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{\eta_r(s)}{\psi_r(s)} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}(s)^{\ell\times k}$$

is the Smith-McMillan form of $\hat{G}(s)$. To proceed we should consider separately the square and nonsquare cases.

Square transfer matrix When $\hat{G}(s)$ is square and full rank (i.e., $\ell = k = r$),

$$\det \hat{G}(-s)'\hat{G}(s) = \bar{c}\frac{\eta_1(-s)\cdot\eta_k(-s)\,\eta_1(s)\cdot\eta_k(s)}{\psi_1(-s)\cdot\psi_k(-s)\,\psi_1(s)\cdot\psi_k(s)} = \bar{c}\frac{z_G(-s)z_G(s)}{p_G(-s)p_G(s)},$$

where $z_G(s)$ and $p_G(s)$ are the zero and pole polynomials of $\hat{G}(s)$, respectively, and $\bar{c}$ is the (constant) product of the determinants of all the unimodular matrices. When the realization is minimal, $p_G(s) = \Delta(s)$ and (3.8) simplifies to

$$\det(sI - \mathbf{H}) \to c\,\bar{c}\,z_G(s)z_G(-s).$$

Two conclusions can be drawn from here:

1. When $\hat{G}(s)$ has q transmission zeros

$$a_i + jb_i, \quad i \in \{1, 2, \dots, q\},$$

then $2q$ of the eigenvalues of $\mathbf{H}$ converge to

$$\pm a_i \pm jb_i, \quad i \in \{1, 2, \dots, q\},$$

and therefore q closed-loop poles converge to

$$-|a_i| + jb_i, \quad i \in \{1, 2, \dots, q\}.$$

Sidebar 13: Recall that the poles of the closed-loop are only the stable eigenvalues of $\mathbf{H}$, which converge to both $a_i + jb_i$ and $-a_i - jb_i$.

2. When the transfer function $\hat{G}(s)$ from the control input u to the controlled output z does not have any transmission zero, $\mathbf{H}$ has no finite eigenvalues as $\rho \to 0$ and therefore all closed-loop poles must converge to infinity.

Nonsquare transfer matrix When $\hat{G}(s)$ is not square and/or not full-rank, replacing (3.8) in (3.7), we obtain

$$\det(sI-\mathbf{H}) \to \bar{c}\Delta(s)\Delta(-s)\det \begin{bmatrix} \frac{\eta_1(-s)}{\psi_1(-s)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\eta_r(-s)}{\psi_r(-s)} \end{bmatrix} L_r(-s)'L_r(s) \begin{bmatrix} \frac{\eta_1(s)}{\psi_1(s)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\eta_r(s)}{\psi_r(s)} \end{bmatrix},$$

where $L_r(s) \in \mathbb{R}[s]^{\ell \times r}$ contains the left-most r columns of $L(s)$. In this case, when the realization is minimal we obtain

$$\det(sI - \mathbf{H}) \to \bar{c}\, z_G(s) z_G(-s) \det L_r(-s)' L_r(s),$$

which shows that for nonsquare matrices $\det(sI - \mathbf{H})$ generally has more roots than the transmission zeros of $\hat{G}(s)$. In this case one needs to compute the stable roots of

$$\Delta(s)\Delta(-s) \det \hat{G}(-s)'\hat{G}(s)$$

to determine the asymptotic locations of the closed-loop poles.

Attention! This means that in general one wants to avoid transmission zeros from the control input u to the controlled output z, especially slow transmission zeros that will attract the poles of the closed-loop. For nonsquare systems one must pay attention to all the zeros of $\det \hat{G}(-s)'\hat{G}(s)$. □

Sidebar 14: This property of LQR, resembles a similar property of the root locus, except that now we have the freedom to choose the controlled output to avoid problematic zeros.

Exercise 7. Show that (3.6) holds.

Hint: Use the following properties of the determinant:

$$\det \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} = \det M_1 \det M_4 \det \big(I - M_3 M_1^{-1} M_2 M_4^{-1}\big) \tag{3.9}$$

$$\det(I + XY) = \det(I + YX) \tag{3.10}$$

(cf. , e.g., [4, Equation 1-235] and [4, Equation 1-201]). □

Solution to Exercise 7. To compute the characteristic polynomial of $\mathbf{H}$ we write

$$\begin{aligned} \det(sI - \mathbf{H}) &= \det \begin{bmatrix} sI - A & BR^{-1}B' \\ G'G & sI + A' \end{bmatrix} \\ &= \det(sI - A)\det(sI + A')\det\Big(I - G'G(sI - A)^{-1}BR^{-1}B'(sI + A')^{-1}\Big) \\ &= \det(sI - A)\det(sI + A')\det\Big(I - R^{-1}B'(sI + A')^{-1}G'G(sI - A)^{-1}B\Big), \end{aligned} \tag{3.11}$$

where we first used (3.9) and then (3.10). Since

$$\det(sI + A') = \det(sI + A) = (-1)^n \det(-sI - A) = (-1)^n \Delta(-s)$$

and

$$\hat{G}(-s)'\hat{G}(s) = -B'(sI + A')^{-1}G'G(sI - A)^{-1}B + H'H,$$

Sidebar 15: Recall that we are assuming that $G'H = 0$.

we can re-write (3.11) compactly as (3.6) ■

3.4.2 Cost

We saw in Chapter #1 that the minimum value of the LQR criterion is given by

$$J_{\text{LQR}} := \int_0^\infty \|z(t)\|^2 + \rho\|u(t)\|^2 \, dt = x'(0)P_\rho x(0).$$

where ρ is positive constant and P_ρ the corresponding solutions to the ARE

$$A'P_\rho + P_\rho A + G'G - P_\rho B R_\rho^{-1} B' P_\rho = 0, \qquad R_\rho := H'H + \rho I, \tag{3.12}$$

Sidebar 16: Here, we use the subscript ρ to emphasize that the solution to the ARE, depends on this parameter.

The following result make explicit the dependence of P_ρ on ρ as this parameter converges to zero:

Sidebar 17: This result was taken from [4, Section 3.8.3, pp. 306–312, cf. Theorem 3.14]. A simple proof for the SISO case can be found in [5, Section 3.5.2, pp. 145–146].

Theorem 4. *When $H = 0$ the solution to (3.12) satisfies:*

$$\lim_{\rho\to 0} P_\rho \begin{cases} = 0 & \ell = k \text{ and all transmission zeros of } \hat{G}(s) \text{ have negative or zero real parts} \\ \neq 0 & \ell = k \text{ and } \hat{G}(s) \text{ has transmission zeros with positive real part} \\ \neq 0 & \ell > k \end{cases}$$

Attention! This result shows a fundamental limitation due to "unstable" transmission zeros. It shows that when there are transmission zeros from the input u to the controlled output z, it is not possible to reduce the energy of z arbitrarily, even if one is willing to spend much control energy. □

Attention! Suppose that $\ell = k$ and and all transmission zeros of $\hat{G}(s)$ have negative or zero real parts. Taking limits on both sides of (3.12) and using the fact that $\lim_{\rho\to 0} P_\rho = 0$, we conclude that

$$\lim_{\rho\to 0} \frac{1}{\rho} P_\rho B B' P_\rho = \lim_{\rho\to 0} \rho K'_\rho K_\rho = G'G,$$

where $K_\rho := R_\rho^{-1} B' P_\rho$ is the state-feedback gain. Assuming that G is full row rank, this implies that

Notation 7: A square matrix S is *orthogonal* if $S'S = SS' = I$.

$$\lim_{\rho\to 0} \sqrt{\rho} K_\rho = SG,$$

Exercise 8: Show that given two matrices $X, M \in \mathbb{R}^{n\times\ell}$ with M full row rank and $X'X = M'M$, there exists an orthogonal matrix $S \in \mathbb{R}^{\ell\times\ell}$ such that $M = SX$.

for some orthogonal matrix S. This shows that asymptotically we have

$$K_\rho = \frac{1}{\sqrt{\rho}} SG,$$

and therefore the optimal control is of the form

$$u = K_\rho x = \frac{1}{\sqrt{\rho}} SGx = \frac{1}{\sqrt{\rho}} Sz,$$

i.e., for these systems *the cheap control problem corresponds to high-gain static feedback of the controlled output.* □

Solution to Exercise 8. Left-multiplying $X'X = M'M$ by $(MM')^{-1}M$, we conclude that $SX = M$ with $S := (MM')^{-1}MX'$. Note that the inverse exists because M is full row rank. Moreover, $SS' = (MM')^{-1}MX'XM'(MM')^{-1} = I$. □

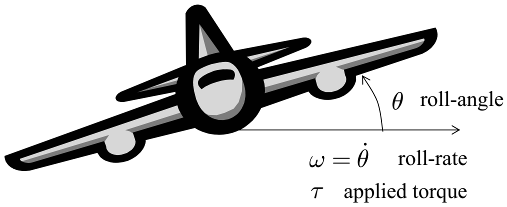
3.5 LQR design example

Example 1 (Aircraft roll-dynamics). Figure 3.3 shows the roll-angle dynamics of an aircraft [6, p. 381]. Defining

$$x := \begin{bmatrix} \theta & \omega & \tau \end{bmatrix}'$$

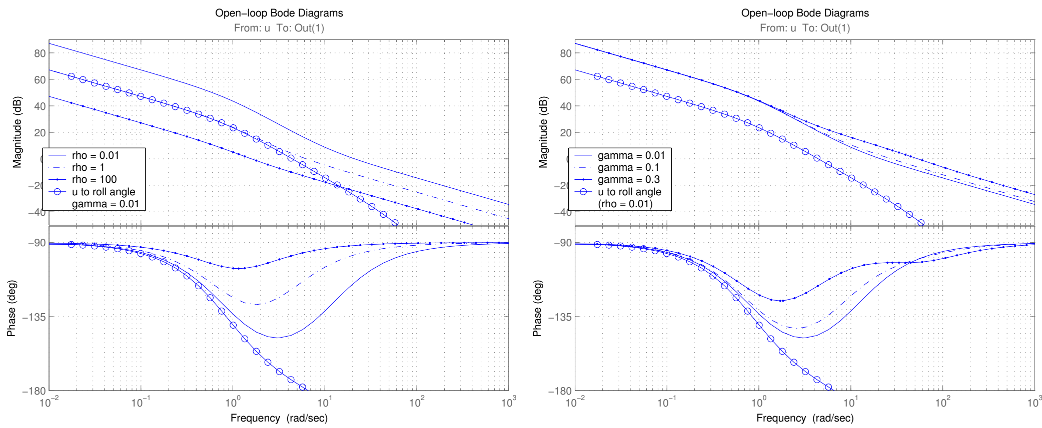
we conclude that

$$\dot{x} = Ax + Bu$$



$$\dot{\theta} = \omega$$
$$\dot{\omega} = -.875\omega - 20\tau$$
$$\dot{\tau} = -50\tau + 50u$$

Figure 3.3. Aircraft roll-angle dynamics



(a) (b)

Figure 3.4. Bode plots for the open-loop gain

with

$$A := \begin{bmatrix} 0 & 1 & 0 \\ 0 & -.875 & -20 \\ 0 & 0 & -50 \end{bmatrix}, \qquad B := \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}.$$

If we have both θ and ω available for control, we can define

$$y := \begin{bmatrix} \theta & \omega \end{bmatrix}' = Cx + Du$$

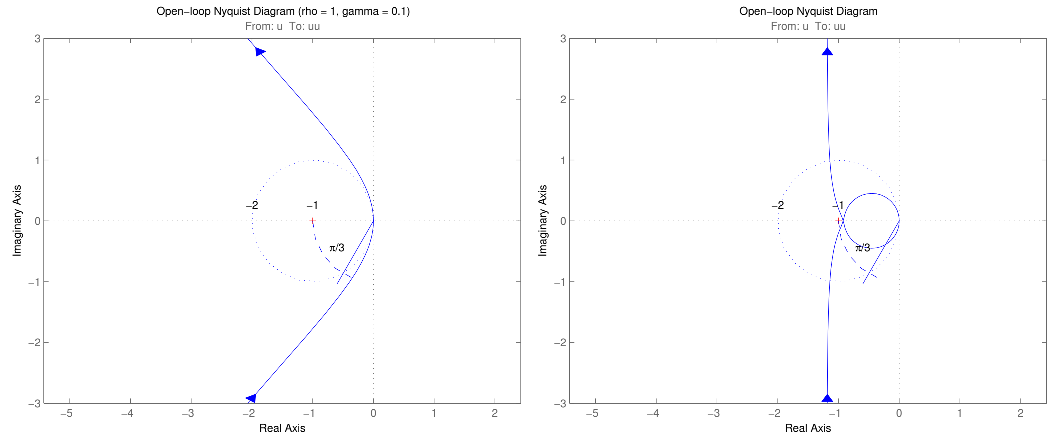
with

$$C := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad D := \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Open-loop gains Figure 3.4 shows Bode plots of the open-loop gain $\hat{L}(s) = K(sI - A)^{-1}B$ for several LQR controllers obtained for this system. The controlled output was chosen to be $z := \begin{bmatrix} \theta & \gamma\dot{\theta} \end{bmatrix}'$, which corresponds to

$$G := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \end{bmatrix}, \qquad H := \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The controllers were obtained with $R = 1$, $Q = I_{2\times 2}$, and several values for ρ and γ.



(a) (b)

Figure 3.5. Nyquist plots for the open-loop gain

Figure 3.4(a) shows the open-loop gain for several values of ρ, where we can see that ρ allow us to move the whole magnitude Bode plot up and down. Figure 3.4(b) shows the open-loop gain for several values of γ, where we can see that a larger γ results in a larger phase margin. As expected, for low frequencies the open-loop gain magnitude matches that of the process transfer function from u to θ (but with significantly lower/better phase) and at high-frequencies the gain's magnitude falls at -20dB/decade.

Figure 3.5 shows Nyquist plots of the open-loop gain $\hat{L}(s) = K(sI - A)^{-1}B$ for different choices of the controlled output z. In Figure 3.5(a) $z := \begin{bmatrix}\theta & \dot{\theta}\end{bmatrix}'$, which corresponds to

$$G := \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, \qquad H := \begin{bmatrix}0\\ 0\end{bmatrix}.$$

In this case, $H'G = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$ and Kalman's inequality holds as can be seen in the Nyquist plot. In Figure 3.5(b), the controlled output was chosen to be $z := \begin{bmatrix}\theta & \dot{\tau}\end{bmatrix}'$, which corresponds to

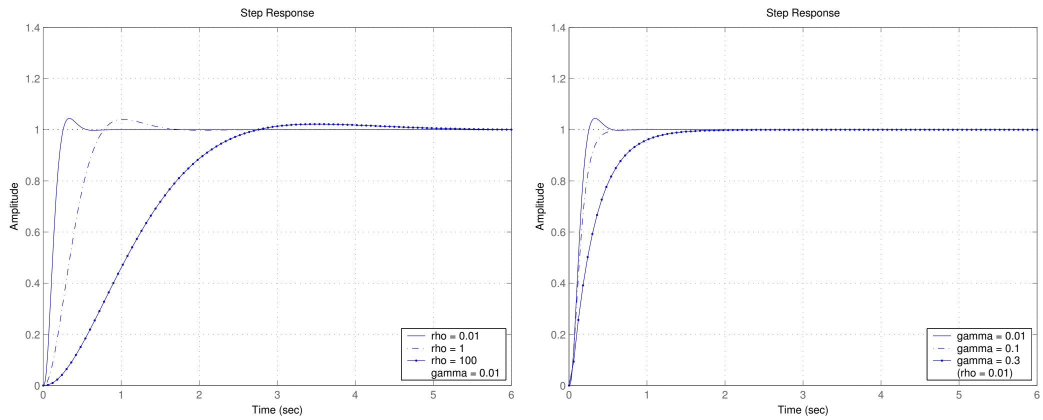
$$G := \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -50\end{bmatrix}, \qquad H := \begin{bmatrix}0\\ 50\end{bmatrix}.$$

In this case we have $H'G = \begin{bmatrix}0 & 0 & -2500\end{bmatrix}$ and Kalman's inequality does not hold. We can see from the Nyquist plot that the phase and gain margins are very small and there is little robustness with respect to unmodeled dynamics since a small perturbation in the process can lead to an encirclement of the point -1.

Step responses Figure 3.6 shows step responses for the state-feedback LQR controllers whose Bode plots for the open-loop gain are shown in Figure 3.4. Figure 3.6(a) shows that smaller values of ρ lead to faster responses and Figure 3.6(b) shows that larger values for γ lead to smaller overshoots (but slower responses).

3.6 Matlab commands

Matlab Hint 2 (`sigma`). The command `sigma(sys)` draws the norm-Bode plot of the system `sys`. For scalar transfer functions this command plots the usual magnitude Bode



(a) (b)

Figure 3.6. Closed-loop step responses

plot but for vector transfer function it plots the norm of the transfer function versus the frequency. □

Matlab Hint 3 (`nyquist`). The command `nyquist(sys)` draws the Nyquist plot of the system `sys`.

Especially when there are poles very close to the imaginary axis (e.g., because they were actually on the axis and you moved them slightly to the left), the automatic scale may not be very good because it may be hard to distinguish the point -1 from the origin. In this case, you can use then zoom features of MATLAB to see what is going on near -1: Try clicking on the magnifying glass and selecting a region of interest; or try left-clicking on the mouse and selecting "zoom on (-1,0)" (without the magnifying glass selected.) □

3.7 Additional notes

Sidebar 10 (Multi-variable Nyquist Criterion). The Nyquist criterion is used to investigate the stability of the negative feedback connection in Figure 3.7. It allows one to compute the number of unstable (i.e., in the closed right-hand-side plane) poles of the closed-loop transfer matrix $\big(I + \hat{G}(s)\big)^{-1}$ as a function of the number of unstable poles of the open-loop transfer function $\hat{G}(s)$.

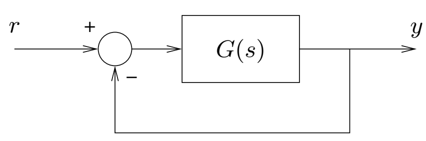


Figure 3.7. Negative feedback

To apply the criterion we start by drawing the *Nyquist plot of* $\hat{G}(s)$, which is done by evaluating $\det\big(I + \hat{G}(j\omega)\big)$ from $\omega = -\infty$ to $\omega = +\infty$ and plotting it in the complex plane. This leads to a closed-curve that is always symmetric with respect to the real axis. This curve should be annotated with arrows indicating the direction corresponding to increasing ω.

Matlab hint 3: `nyquist(sys)` draws the Nyquist plot of the system `sys`...

Sidebar 18: The Nyquist plot should be viewed as the image of a clockwise contour that goes along the axis and closes with a right-hand-side loop at ∞.

Any poles of $\hat{G}(s)$ on the imaginary axis should be moved slightly to the left of the axis because the criterion is only valid when $\hat{G}(s)$ is analytic on the imaginary axis. E.g.,

$$\hat{G}(s) = \frac{s+1}{s(s-3)} \qquad \longrightarrow \qquad \hat{G}_\epsilon(s) \approx \frac{s+1}{(s+\epsilon)(s-3)}$$

$$\hat{G}(s) = \frac{s}{s^2+4} = \frac{s}{(s+2j)(s-2j)} \qquad \longrightarrow \qquad \hat{G}_\epsilon(s) \approx \frac{s}{(s+\epsilon+2j)(s+\epsilon-2j)} = \frac{s}{(s+\epsilon)^2+4},$$

for a small $\epsilon > 0$. The criterion should then be applied to the "perturbed" transfer function $\hat{G}_\epsilon(s)$. If we conclude that the closed-loop is asymptotically stable for $\hat{G}_\epsilon(s)$ with very small $\epsilon > 0$, then the closed-loop with $\hat{G}(s)$ will also be asymptotically stable and vice-versa.

Sidebar 19: To compute #ENC, we draw a ray from the origin to ∞ in *any* direction and add one each time the Nyquist plot crosses the ray in the clockwise direction (with respect to the origin of the ray) and subtract one each time it crosses the ray in the counter-clockwise direction. The final count gives #ENC.

Attention! For the multi-variable Nyquist criteria we count encirclements around the origin and not around -1, because the multi-variable Nyquist plot is "shifted" to the right by adding the I to in det $\big(I + \hat{G}(j\omega)\big)$.

Nyquist Stability Criterion. *The total number of* unstable *(closed-loop) poles of* $\big(I + \hat{G}(s)\big)^{-1}$ *(*#CUP*) is given by*

$$\#\text{CUP} = \#\text{ENC} + \#\text{OUP},$$

where #OUP *denotes the number of unstable (open-loop) poles of* $\hat{G}(s)$ *and* #ENC *the number of clockwise encirclements by the multi-variable Nyquist plot around the origin. To have a stable closed-loop one thus needs*

$$\#\text{ENC} = -\#\text{OUP}. \qquad \square$$

Chapter 4

Output-feedback

Summary

1. Deterministic minimum-energy estimation (MEE)
2. Stochastic Linear Quadratic Gaussian (LQG) estimation
3. LQG/LQR output feedback
4. Loop transfer recovery (LTR) with design example
5. Optimal set-point control

4.1 Certainty equivalence

The state-feedback LQR formulation considered in Chapter 1 suffered from the drawback that the optimal control law

$$u(t) = -Kx(t) \tag{4.1}$$

required the whole state x of the process to be measurable. A possible approach to overcome this difficulty is to construct an estimate $\hat{x}$ of the state of the process based solely on the past values of the measured output y and control signal u, and then use

$$u(t) = -K\hat{x}(t)$$

instead of (4.1). This approach is usually known as *certainty equivalence* and leads to the architecture in Figure 4.1. In this chapter we consider the problem of constructing state

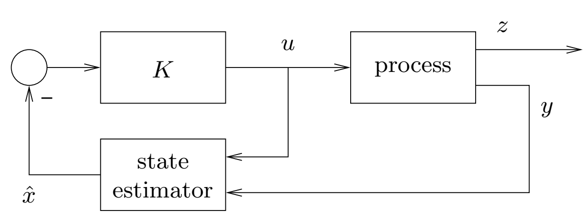


Figure 4.1. Certainty equivalence controller

estimates for use in certainty equivalence controllers.

4.2 Deterministic Minimum-Energy Estimation (MEE)

Consider a continuous-time LTI system of the form:

$$\dot{x} = Ax + Bu, \qquad y = Cx, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ y \in \mathbb{R}^m, \tag{CLTI}$$

where u is the control signals and y the measured output. Estimating the state x at some time t can be viewed as solving (CLTI) for the unknown x, for given $u(\tau), y(\tau)$, $\tau \leq t$.

Assuming that the model (CLTI) is exact and observable, $x(t)$ could be reconstructed exactly using the constructibility Gramian

$$x(t) = W_{Cn}(t_0,t)^{-1}\Big(\int_{t_0}^{t} e^{A'(\tau-t)}C'y(\tau)d\tau + \int_{t_0}^{t}\int_{\tau}^{t} e^{A'(\tau-t)}C'Ce^{A(\tau-s)}Bu(s)dsd\tau\Big),$$

where

$$W_{Cn}(t_0,t) := \int_{t_0}^{t} e^{A'(\tau-t)}C'Ce^{A(\tau-t)}d\tau$$

In practice, the model (CLTI) is never exact and the measured output y is generated by a system of the following form

$$\dot{x} = Ax + Bu + \bar{B}d, \qquad y = Cx + n, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ d \in \mathbb{R}^q,\ y \in \mathbb{R}^m, \tag{4.2}$$

where d represents a disturbance and n measurement noise. Since neither d nor n are know, solving (4.2) for x no longer yields a unique solution since essentially any state value could explain the measured output for sufficiently large noise/disturbances.

Minimum-Energy Estimation (MEE) consists of finding the state trajectory

$$\dot{\bar{x}} = A\bar{x} + Bu + \bar{B}d, \qquad y = C\bar{x} + n, \qquad \bar{x} \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ d \in \mathbb{R}^q,\ y \in \mathbb{R}^m \tag{4.3}$$

that is consistent with the past measured output y and control signal u for the *least amount of noise n and disturbance d,* measured by

$$J_{\text{MEE}} := \int_{-\infty}^{t} n(\tau)'Qn(\tau) + d(\tau)'Rd(\tau)d\tau \tag{4.4}$$

where $Q \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{q\times q}$ are symmetric positive definite matrices. Once this trajectory has been found based on the data collected on the interval $(\infty, t]$, the minimum-energy state estimate is simply the most recent value of $\bar{x}$:

$$\hat{x}(t) = \bar{x}(t).$$

The role of the matrices Q and R can be understood as follows:

1. When we choose Q large, we are forcing the noise term to be small, which means that we "believe" in the measured output. This leads to state estimators that respond fast to changes in the output y.

2. When we choose R large, we are forcing the disturbance term to be small, which means that we "believe" in the past values of the state-estimate and will respond cautiously (slowly) to unexpected changes in the measured output.

4.2.1 Solution to the MEE problem

The MEE problem is solved by minimizing the quadratic cost

$$J_{\text{MEE}} = \int_{-\infty}^{t} \big(C\bar{x}(\tau) - y(\tau)\big)' Q \big(C\bar{x}(\tau) - y(\tau)\big) + d(\tau)' R d(\tau) d\tau$$

for the system (4.3). Like in the LQR problem, this can be done by square completion:

Theorem 5 (Minimum-Energy Estimation). *Assume that there exists a symmetric positive definite solution P to the following ARE*

$$(-A')P + P(-A) + C'QC - P\bar{B}R^{-1}\bar{B}'P = 0. \tag{4.5}$$

for which $-A - \bar{B}R^{-1}\bar{B}'P$ is Hurwitz. Then the MME estimator for (4.2) *for the criteria* (4.4) *is given by*

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly, \qquad L := P^{-1}C'Q. \tag{4.6}$$

4.2.2 Dual Algebraic Riccati Equation

To determine conditions for the existence of an appropriate solution to the ARE (4.5), it is convenient to left- and right-multiplying this equation by $S := P^{-1}$ and then multiplying it -1. This procedure yields an equivalent equation called the *dual Algebraic Riccati Equation*:

$$AS + SA' + \bar{B}R^{-1}\bar{B}' - SC'QCS = 0. \tag{4.7}$$

The gain L can be written in terms of the solution S to the dual ARE as $L := SC'Q$.

To solve the MEE problem one needs to find a symmetric positive definite solution to the dual ARE for which $-A - \bar{B}R^{-1}\bar{B}'S^{-1}$ is Hurwitz. The results in Chapter 2 provide conditions for the existence of an appropriate solution to the dual ARE (4.5):

Theorem 6. *Assume that the pair $(A, \bar{B})$ is controllable and that the pair (A, C) is detectable.*

(i) There exists a symmetric positive definite solution S to the dual ARE (4.7)*, for which $A - LC$ is Hurwitz.*

(ii) There exists a symmetric positive definite solution $P := S^{-1}$ to the ARE (4.5)*, for which $-A - \bar{B}R^{-1}\bar{B}'P$ is Hurwitz.* □

Proof of Theorem 6. Part (i) is a straightforward application of Theorem 3 for $N = 0$, and the following facts

1. stabilizability of (A', C') is equivalent to the detectability of (A, C);
2. observability of $(A', \bar{B}')$ is equivalent to the controllability of $(A, \bar{B})$;
3. $A' - C'L'$ is Hurwitz if and only if $A + LC$ is Hurwitz.

The fact that $P := S^{-1}$ satisfies (4.5) has already been established from the construction of the dual ARE (4.7). To prove (ii) it remains to show that $-A - \bar{B}R^{-1}\bar{B}'P$ is Hurwitz. To this effect we re-write (4.5) as

$$(-A' - P\bar{B}R^{-1}\bar{B}')P + P(-A - \bar{B}R^{-1}\bar{B}'P) = -S, \qquad S := C'QC + P\bar{B}R^{-1}\bar{B}'P.$$

The stability of $-A - \bar{B}R^{-1}\bar{B}'P$ then follows from the controllability Lyapunov test because the pair $(-A - \bar{B}R^{-1}\bar{B}'P, S)$ is controllable .

■ **Exercise 9:** Verify that this is so using the eigenvector test.

4.2.3 Convergence of the estimates

The MEE estimator is often written as

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \qquad \hat{y} = C\hat{x} \qquad L := SC'Q. \tag{4.8}$$

Defining the state estimation error $e = x - \hat{x}$, we conclude from (4.8) and (4.2) that

$$\dot{e} = (A - LC)e + \bar{B}d - Ln.$$

Since $A - LC$ is Hurwitz, we conclude that, in the absence of measurement noise and disturbances, $e(t) \to 0$ as $t \to \infty$ and therefore $\|x(t) - \hat{x}(t)\| \to 0$ as $t \to \infty$.

Sidebar 20: Why? Because the poles of the transfer functions from d and n to e are the eigenvalues of $A - LC$.

In the presence of noise we have BIBO stability from the inputs d and n to the "output" e so $\hat{x}(t)$ may not converge to $x(t)$ but at least does not diverge from it.

4.2.4 Proof of the MEE Theorem

Due to the exogenous term $y(\tau)$ in the MEE criteria, we need more sophisticated invariants so solve this problem. Let P be a symmetric matrix and $\beta : (-\infty, t] \to \mathbb{R}^n$ a differentiable signal to be selected shortly. Assuming that $\bar{x}(t) - \beta(t) \to 0$ as $t \to -\infty$, we obtain

$$\int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau}\big((\bar{x} - \beta)'P(\bar{x} - \beta)\big)d\tau = \big(\bar{x}(t) - \beta(t)\big)'P\big(\bar{x}(t) - \beta(t)\big).$$

Expanding the left-hand side, we conclude that

$$\begin{aligned}\int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau}\big((\bar{x} - \beta)'P(\bar{x} - \beta)\big)d\tau = \int_{-\infty}^{t} &\bar{x}(A'P + PA)\bar{x} + 2\bar{x}(PBu + P\bar{B}d - P\dot{\beta} - A'P\beta) \\ &+2\beta'P(\dot{\beta} - Bu - \bar{B}d)\, d\tau = \big(\bar{x}(t) - \beta(t)\big)'P\big(\bar{x}(t) - \beta(t)\big).\end{aligned}$$

Using this, we can write

$$\begin{aligned}J_{\text{MEE}} = (\bar{x} - \beta)'P(\bar{x} - \beta) + \int_{-\infty}^{t} &\bar{x}(-A'P - PA + C'QC)\bar{x} \\ - 2\bar{x}'(PBu + P\bar{B}d - P\dot{\beta} - A'P\beta + C'Qy) &- 2\beta'P(\dot{\beta} - Bu - \bar{B}d) + y'Qy + d'Rd\, d\tau\end{aligned}$$

To carry out the square-completion argument we combine all the terms in d into a single quadratic form:

$$\begin{aligned}- 2(\bar{x}' - \beta')P\bar{B}d + d'Rd = (d' - (\bar{x}' - \beta)P\bar{B}R^{-1})R(d - R^{-1}\bar{B}'P(\bar{x} - \beta)) \\ - \bar{x}P\bar{B}R^{-1}\bar{B}'P\bar{x} - \beta P\bar{B}R^{-1}\bar{B}'P\beta + 2\bar{x}'P\bar{B}R^{-1}\bar{B}'P\beta.\end{aligned}$$

Replacing this in J_{MEE}, we conclude that

$$\begin{aligned}J_{\text{MEE}} = (\bar{x} - \beta)'P(\bar{x} - \beta) + \int_{-\infty}^{t} &\bar{x}\big(- A'P - PA + C'QC - P\bar{B}R^{-1}\bar{B}'P\big)\bar{x} \\ &- 2\bar{x}'\big(PBu - P\dot{\beta} - (A'P + P\bar{B}R^{-1}\bar{B}'P)\beta + C'Qy\big) \\ &- 2\beta'P(\dot{\beta} - Bu) - \beta P\bar{B}R^{-1}\bar{B}'P\beta + y'Qy \\ &+ (d' - (\bar{x}' - \beta)P\bar{B}R^{-1})R(d - R^{-1}\bar{B}'P(\bar{x} - \beta))\, d\tau.\end{aligned}$$

Picking P to be the solution to the ARE (4.5) and the signal β to satisfy

$$P\dot{\beta} = -(A'P + P\bar{B}R^{-1}\bar{B}'P)\beta + PBu + C'Qy = 0$$

$$\Leftrightarrow \quad \dot{\beta} = (A - P^{-1}C'QC)\beta + Bu + P^{-1}C'Qy, \tag{4.9}$$

we obtain

$$J_{\text{MEE}} = J_0 + (\bar{x} - \beta)'P(\bar{x} - \beta) + \int_{-\infty}^{t} (d' - (\bar{x}' - \beta)P\bar{B}R^{-1})R(d - R^{-1}\bar{B}'P(\bar{x} - \beta))\, d\tau,$$

where

$$J_0 := \int_{-\infty}^{t} -2\beta'P(\dot{\beta} - Bu) - \beta P\bar{B}R^{-1}\bar{B}'P\beta + y'Qy.$$

This criteria is minimized by setting

$$\bar{x}(t) = \beta(t), \qquad d(\tau) = R^{-1}\bar{B}'P\big(\bar{x}(\tau) + \beta(\tau)\big), \quad \forall \tau < t,$$

which, together with the differential equation (4.3), completely define the optimal trajectory $\bar{x}(\tau)$, $\tau \le t$ that minimizes J_{MEE}. Moreover, (4.9) computes exactly the MEE $\hat{x}(t) = \bar{x}(t) = \beta(t)$. Note that under this choice of d, we have

$$(\dot{\bar{x}} - \dot{\beta}) = (A + \bar{B}R^{-1}\bar{B}'P)(\bar{x} - \beta) + P^{-1}C'Q(C\beta - y).$$

Therefore $\bar{x} - \beta \to 0$ as $t \to -\infty$ (assuming that $u, y, \beta \to 0$ as $t \to -\infty$) because $-A - \bar{B}R^{-1}\bar{B}'P$ is asymptotically stable. ■

4.3 Linear Quadratic Gaussian (LQG) estimation

The MEE introduced before also has a stochastic interpretation. To state it, we consider again the continuous-time LTI system

$$\dot{x} = Ax + Bu + \bar{B}d, \qquad y = Cx + n, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ d \in \mathbb{R}^q,\ y \in \mathbb{R}^m,$$

but now assume that the disturbance d and the measurement noise n are uncorrelated zero-mean Gaussian white noise stochastic processes with co-variance matrices

$$\text{E}[d(t)d'(\tau)] = \delta(t - \tau)R^{-1}, \qquad \text{E}[n(t)n'(\tau)] = \delta(t - \tau)Q^{-1}, \qquad R, Q > 0.$$

The MEE state estimate $\hat{x}(t)$ given by equation (4.6) in Section 4.2 also minimizes the asymptotic norm of the estimation error

$$J_{\text{LQG}} := \lim_{t \to \infty} \text{E}[\|x(t) - \hat{x}(t)\|^2]$$

This is consistent with what we saw before regarding the roles of the matrices Q and R in MEE:

Sidebar 21: In this context, the estimator (4.6) is usually called a *Kalman filter*.

Matlab hint 4: `kalman` computes the optimal MEE/LQG estimator gain L.

1. A large Q corresponds to little measurement noise and leads to state estimators that respond fast to changes in the measured output.

2. A large R corresponds to small disturbances and leads to state-estimates that respond cautiously (slowly) to unexpected changes in the measured output.

4.4 LQR/LQG output feedback

We now go back to the problem of designing an output-feedback controller for the following continuous-time LTI process:

$$\begin{aligned} \dot{x} &= Ax + Bu + \bar{B}d, && x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ d \in \mathbb{R}^q, \\ y &= Cx + n, && y, n \in \mathbb{R}^m, \\ z &= Gx + Hu, && z \in \mathbb{R}^\ell \end{aligned}$$

Suppose that we designed a state-feedback controller

$$u = -Kx \tag{4.11}$$

that solves an LQR problem and constructed an LQG/MEE state-estimator

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly.$$

Matlab hint 5: `reg(sys,K,L)` computes the LQG/LQR *positive* output-feedback controller for the process `sys` with regulator gain `K` and estimator gain `L`

We can obtain an output-feedback controller by using the estimated state $\hat{x}$ in (4.11), instead of the true state x. This leads to the following output-feedback controller

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly = (A - LC - BK)\hat{x} + Ly, \qquad u = -K\hat{x},$$

with *negative-feedback* transfer matrix given by

$$\hat{C}(s) = K(sI - A + LC + BK)^{-1}L.$$

This is usually known as an *LQG/LQR output-feedback controller*. Since both $A - BK$ and $A - LC$ are asymptotically stable, the separation principle guarantees that this controller makes the closed-loop system asymptotically stable.

Exercise 10. Verify that the LQG/LQR controller makes the closed-loop asymptotically stable. □

Solution to Exercise 10. To verify that an LQG/LQR controller makes the closed-loop asymptotically stable, we collect all the equations that define the closed-loop system:

$$\begin{aligned} \dot{x} &= Ax + Bu + \bar{B}d, && y = Cx + n, && z = Gx + Hu \\ \dot{\hat{x}} &= (A - LC)\hat{x} + Bu + Ly, && u = -K\hat{x}. \end{aligned}$$

To check the stability of this system it is more convenient to consider the dynamics of the estimation error $e := x - \hat{x}$ instead of the the state estimate $\hat{x}$. To this effect we replace in the above equations $\hat{x}$ by $x - e$, which yields:

$$\begin{aligned} \dot{x} &= Ax + Bu + \bar{B}d = (A - BK)x + BKe + \bar{B}d, && z = (G - HK)x + HKe, \\ \dot{e} &= (A - LC)e + \bar{B}d - Ln, && u = -K(x - e). \end{aligned}$$

This can be written in matrix notation as

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} \bar{B} & 0 \\ \bar{B} & -L \end{bmatrix} \begin{bmatrix} d \\ n \end{bmatrix}, \qquad z = \begin{bmatrix} G - HK & HK \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}.$$

Stability of the closed loop follows from the triangular structure of this system and the fact that the matrices $A - BK$ and $A - LC$ are both Hurwitz. ■

4.5 Loop transfer recovery (LTR)

We saw in Chapter #3 that a state-feedback controller

$$u = -Kx$$

for the process (4.10) has desirable robustness properties and that we can even shape its open-loop gain

$$\hat{L}(s) = K(sI - A)^{-1}B$$

by appropriate choice of the LQR weighting parameter ρ and the controlled output z.

Suppose now that the state is not accessible and that we constructed an LQG/LQR output-feedback controller with *negative-feedback* transfer matrix given by

$$\hat{C}(s) = K(sI - A + LC + BK)^{-1}L,$$

where $L := SC'Q$ and S is a solution to the dual ARE

$$AS + SA' + \bar{B}R^{-1}\bar{B}' - SC'QCS = 0.$$

for which $A - LC$ is Hurwitz.

In general there is not guarantee that LQG/LQG controllers will inherit the open-loop gain of the original state-feedback design. However, for processes that do not have transmission zeros in the closed right-hand-side complex plane, one can recover the LQR open-loop gain by appropriate design of the state-estimator:

Theorem 7 (Loop transfer recovery). *Suppose that the transfer function*

$$\hat{P}(s) := C(sI - A)^{-1}B$$

from u to y is square ($k = m$) *and has* no transmission zeros in the closed right half-place. *Selecting*

$$\bar{B} := B, \qquad R := I, \qquad Q := \sigma I, \quad \sigma > 0,$$

the open-loop gain for the output-feedback LQG/LQR controller converges to the open-loop gain for the LQR state-feedback controller over a range of frequencies $[0, \omega_{\max}]$ as we make $\sigma \to +\infty$, i.e.,

$$C(j\omega)P(j\omega) \xrightarrow{\sigma \to +\infty} \hat{L}(j\omega), \quad \forall \omega \in [0, \omega_{\max}]. \qquad \square$$

Sidebar 22: $\bar{B} = B$ corresponds to an *input disturbance* since the process becomes $\dot{x} = Ax + B(u + d)$.

Sidebar 23: In general, the larger $\omega_{\max}$ is, the larger σ needs to be for the gains to match.

Attention! 1. To achieve loop-gain recovery we need to chose $Q = \sigma I$, *regardless of the noise statistics.*

2. One should not make σ larger than necessary because we do not want to recover the (slow) -20dB/decade magnitude decrease at high frequencies. In practice we should make σ *just large enough to get loop-recovery until just above or at cross-over.* For larger values of ω, the output-feedback controller may actually behave much better than the state-feedback one.

3. When the process has *zeros in the right half-plane,* loop-gain recovery will generally only work up to the frequencies of the nonminimum-phase zeros.

   When the zeros are in the *left half-plane but close to the axis,* the closed-loop will not be very robust with respect to uncertainty in the position of the zeros. This is because the controller will attempt to cancel these zeros. $\square$

4.6 Optimal set-point

Often one wants the controlled output z to converge as fast as possible to a given nonzero constant *set-point value* r, corresponding to an equilibrium point $(x_{\text{eq}}, u_{\text{eq}})$ of

$$\dot{x} = Ax + Bu, \qquad z = Gx + Hu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ z \in \mathbb{R}^\ell, \tag{4.12}$$

Exercise 11: Show that for $\ell = 1$ we can take $u_{\text{eq}} = 0$, when the matrix A has an eigenvalue at the origin and this mode is observable through z. Note that in this case the process already has an integrator.

Solution: just take x_{eq} to be the corresponding eigenvector, scaled so that $Gx_{\text{eq}} = 1$.

i.e., a pair $(x_{\text{eq}}, u_{\text{eq}})$ for which

$$\begin{cases} Ax_{\text{eq}} + Bu_{\text{eq}} = 0 \\ r = Gx_{\text{eq}} + Hu_{\text{eq}} \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} -A & B \\ -G & H \end{bmatrix}_{(n+\ell)\times(n+k)} \begin{bmatrix} -x_{\text{eq}} \\ u_{\text{eq}} \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}. \tag{4.13}$$

This corresponds to an LQR criterion of the form

$$J_{\text{LQR}} := \int_0^\infty \tilde{z}(t)'\bar{Q}\tilde{z}(t) + \rho\, \tilde{u}'(t)\bar{R}\tilde{u}(t)\, dt, \tag{4.14}$$

where $\tilde{z} := z - r$, $\tilde{u} := u - u_{\text{eq}}$.

To understand when this is possible, three distinct cases should be considered:

1. When the number of inputs k is strictly smaller than the number of controlled outputs ℓ, we have an *under-actuated system*. In this case, the system of equations (4.13) generally does not have a solution because it presents more equations than unknowns.

Attention! This Rosenbrock's matrix is obtained by regarding the controller output as the only output of the system.

2. When the number of inputs k is equal to the number of controlled outputs ℓ, (4.13) always has a solution as long as the Rosenbrock's system matrix

$$P(s) := \begin{bmatrix} sI - A & B \\ -G & H \end{bmatrix}$$

Sidebar 24: Recall that a transmission zero of a transfer matrix is always an invariant zero of its state-space realizations.

is nonsingular for $s = 0$. This means that $s = 0$ should not be an invariant zero of the system and therefore it cannot also be a transmission zero of the transfer function $G(sI - A)^{-1}B + H$.

Intuitively, one should expect problems when $s = 0$ is an invariant zero of the system because when the state converges to an equilibrium point, the control input $u(t)$ must converge to a constant. By the zero-blocking property one should then expect the controlled output $z(t)$ to converge to zero and not to r.

3. When the number of inputs k is strictly larger than the number of controlled outputs ℓ we have an *over-actuated system* and (4.13) generally has multiple solutions.

When $P(0)$ is full row-rank, i.e., when it has $n + \ell$ linearly independent rows, the $(n + \ell) \times (n + \ell)$ matrix $P(0)P(0)'$ is nonsingular and one solution to (4.13) is given by

Exercise 12: Verify that this is so by direct substitution of the "candidate" solution in (4.13).

$$\begin{bmatrix} -x_{\text{eq}} \\ u_{\text{eq}} \end{bmatrix} = P(0)'\big(P(0)P(0)'\big)^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix}$$

Sidebar 25: $P(0)'\big(P(0)P(0)'\big)^{-1}$ is called the pseudo-inverse of $P(0)$.

Also in this case, $s = 0$ should not be an invariant zero of the system because otherwise $P(0)$ cannot be full rank.

4.6.1 State-feedback: Reduction to optimal regulation

The optimal set-point problem can be reduced to that of optimal regulation by considering an auxiliary system with state $\tilde{x} := x - x_{\text{eq}}$, whose dynamics are:

$$\dot{\tilde{x}} = Ax + Bu = A(x - x_{\text{eq}}) + B(u - u_{\text{eq}}) + (Ax_{\text{eq}} + Bu_{\text{eq}})$$
$$\tilde{z} = Gx + Hu - r = G(x - x_{\text{eq}}) + H(u - u_{\text{eq}}) + (Gx_{\text{eq}} + Hu_{\text{eq}} - r)$$

The last terms on each equation cancel because of (4.13) and we obtain

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}, \qquad \tilde{z} = G\tilde{x} + H\tilde{u}. \tag{4.15}$$

We can then regard (4.14) and (4.15) as an *optimal regulation problem* for which the optimal solution is given by

$$\tilde{u}(t) = -K\tilde{x}(t)$$

as in Theorem 1. Going back to the original input and state variables u and x, we conclude that the optimal control for the set-point problem defined by (4.12) and (4.14) is given by

$$u(t) = -K\big(x(t) - x_{\text{eq}}\big) + u_{eq}, \qquad t \geq 0. \tag{4.16}$$

Since the solution to (4.13) can be written in form

$$x_{\text{eq}} = Fr, \qquad u_{\text{eq}} = Nr,$$

for appropriately defined matrices F and N, this corresponds to the control architecture in Figure 4.2.

Sidebar 26: As seen in Exercise 11, the feed-forward term Nr is absent when the process has an integrator.

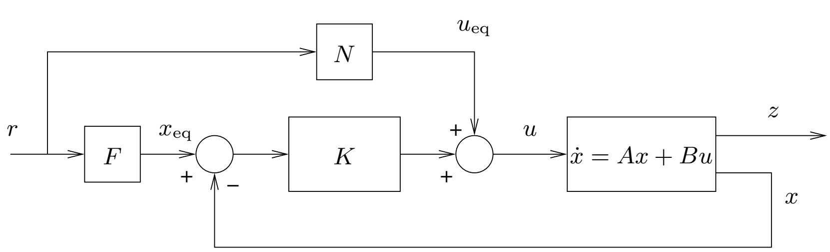


Figure 4.2. Linear quadratic set-point control with state feedback

Closed-loop transfer functions To determine the transfer function from the reference r to the control input u, we use the diagram in Figure 4.2 to conclude that

$$\hat{u} = N\hat{r} + KF\hat{r} - K(sI - A)^{-1}B\hat{u} \quad \Leftrightarrow \quad \hat{u} = \big(I + \hat{L}(s)\big)^{-1}(N + KF)\hat{r},$$

where $\hat{L}(s) := K(sI - A)^{-1}B$ is the open-loop gain of the LQR state-feedback controller. We therefore conclude that

1. When the open-loop gain $\hat{L}(s)$ is small we essentially have

$$\hat{u} \approx (N + KF)\hat{r}.$$

Since at high frequencies $\hat{L}(s)$ falls at -20dB/dec the transfer function from r to u will always converge to $N + KF \neq 0$ at high frequencies.

Sidebar 27: $N + KF$ is always nonzero since otherwise the reference would not affect the control input.

2. When the open-loop gain $\hat{L}(s)$ is large we essentially have

$$\hat{u} \approx \hat{L}(s)^{-1}(N + KF)\hat{r}.$$

To make this transfer function small we necessarily need to increase the open-loop gain $\hat{L}(s)$.

The transfer function from r to the controlled output z can be obtained by composing the transfer function from r to u just computed with the transfer function from u to z:

$$\hat{z} = \hat{G}(s)\big(I + \hat{L}(s)\big)^{-1}(N + KF)\hat{r},$$

where $\hat{G}(s) := G(sI - A)^{-1}B + H$. We therefore conclude that

1. When the open-loop gain $\hat{L}(s)$ is small we essentially have

$$\hat{z} \approx \hat{G}(s)(N + KF)\hat{r},$$

and therefore the closed-loop transfer function mimics that of the process.

2. When the open-loop gain $\hat{L}(s)$ is large we essentially have

$$\hat{z} \approx \hat{G}(s)\hat{L}(s)^{-1}(N + KF)\hat{r}.$$

Moreover, from Kalman's equality we also have that $\|\hat{L}(j\omega)\| \approx \frac{1}{\sqrt{\rho}}\|\hat{G}(j\omega)\|$ when $\|\hat{L}(j\omega)\| \gg 1$, $R := \rho I$, and $H = 0$ (cf. Section 3.3). In this case we obtain

$$\|\hat{z}(j\omega)\| \approx \frac{\|N + KF\|}{\sqrt{\rho}}\|\hat{r}(j\omega)\|,$$

which shows a "flat" Bode plot from r to z.

Sidebar 28: Since z will converge to a constant r, we must have $\|\hat{z}(0)\| = \|\hat{r}(0)\|$ and therefore when $\|\hat{L}(0)\| \gg 1$ we must have $\|N + KF\| \approx \sqrt{\rho}$.

4.6.2 Output-feedback

When the state is not accessible we need to replace (4.16) by

$$u(t) = -K\big(\hat{x}(t) - x_{\text{eq}}\big) + u_{eq}, \qquad t \geq 0,$$

where $\hat{x}$ is the state-estimate produced by an LQG/MEE state-estimator

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly = (A - LC - BK)\hat{x} + BKx_{\text{eq}} + Bu_{eq} + Ly.$$

Defining $\bar{x} := x_{\text{eq}} - \hat{x}$ and using the fact that $Ax_{\text{eq}} + Bu_{\text{eq}} = 0$, we conclude that

$$\dot{\bar{x}} = -(A - LC - BK)\hat{x} + (A - BK)x_{eq} - Ly = (A - LC - BK)\bar{x} - L(y - Cx_{eq}).$$

This allow us to re-write the equations for the *LQG/LQR set-point controller* as:

$$\dot{\bar{x}} = (A - LC - BK)\bar{x} - L(y - Cx_{eq}), \qquad u = K\bar{x} + u_{eq}, \tag{4.17}$$

which corresponds to the control architecture shown in Figure 4.3.

Sidebar 29: When $z = y$, we have $G = C$, $H = 0$ and in this case $Cx_{\text{eq}} = r$. This corresponds to $CF = 1$ in Figure 4.3. When the process has an integrator we get $N = 0$ and obtain the usual unity-feedback configuration.

Exercise 13. Verify that the LQG/LQR set-point controller (4.17) makes the closed-loop asymptotically stable.

Hint: Write the state of the closed-loop in terms of $x - x_{\text{eq}}$ *and* $e := x - \hat{x}$. □

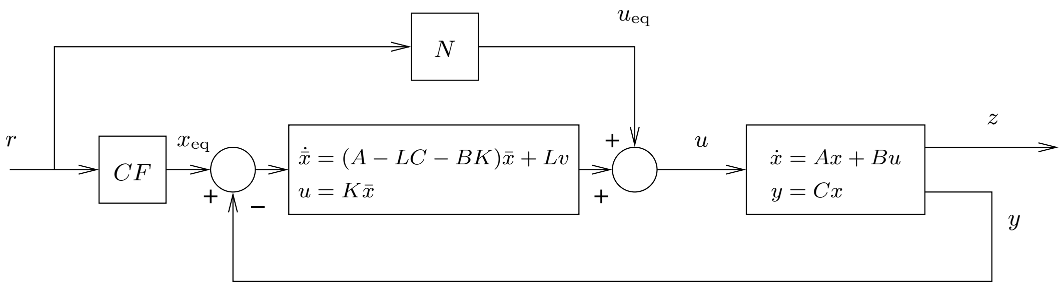


Figure 4.3. LQG/LQR set-point control

Closed-loop transfer functions The closed-loop transfer functions from the reference r to the control input u and controlled output z are now given by

$$\hat{u} = \big(I + \hat{C}(s)\hat{P}(s)\big)^{-1}(N + \hat{C}(s)CF)\hat{r},$$
$$\hat{y} = \hat{G}(s)\big(I + \hat{C}(s)\hat{P}(s)\big)^{-1}(N + \hat{C}(s)CF)\hat{r},$$

where

$$\hat{C}(s) := K(sI - A + LC + BK)^{-1}L, \qquad \hat{P}(s) := C(sI - A)^{-1}B.$$

When LTR succeeds, i.e., when

$$\hat{C}(j\omega)\hat{P}(j\omega) \approx \hat{L}(j\omega), \qquad \forall \omega \in [0, \omega_{\max}],$$

the main difference between these and the formulas seen before for state feedback is th fact that the matrix $N + KF$ multiplying by $\hat{r}$ has been replaced by the transfer matrix $N + \hat{C}(s)CF$.

When $N = 0$, this generally leads to smaller transfer functions when the loop gain is low, because we now have

$$\hat{u} \approx \hat{C}(s)CF\hat{r}, \qquad \hat{y} \approx \hat{G}(s)\hat{C}(s)CF\hat{r},$$

and $\hat{C}(s)$ falls at least at -20dB/dec.

4.7 LQR/LQG with Matlab

Matlab Hint 4 (`kalman`). The command `[est,L,P]=kalman(sys,QN,RN)` computes the optimal LQG estimator gain for the process

$$\dot{x} = \mathtt{A}x + \mathtt{B}u + \mathtt{BB}d, \qquad y = \mathtt{C}x + n,$$

where $d(t)$ and $n(t)$ are uncorrelated zero-mean Gaussian noise processes with co-variance matrices

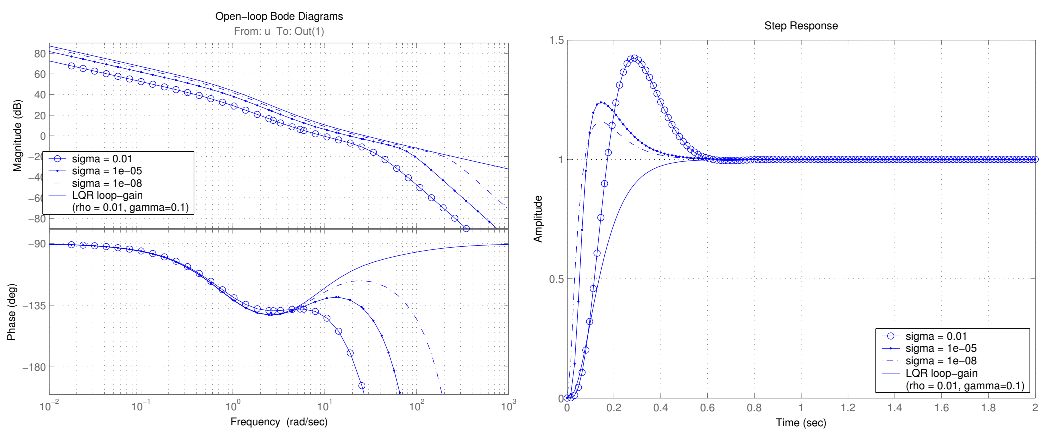
$$\mathrm{E}\left[d\, d'\right] = \mathtt{QN}, \qquad \mathrm{E}\left[n\, n'\right] = \mathtt{RN}.$$

`sys` should be a state-space model defined by `sys=ss(A,[B BB],C,0)`. This command returns the optimal estimator gain `L`, the solution `P` to the corresponding Algebraic Riccati Equation, and a state-space model `est` for the estimator. The inputs to `est` are $[u; y]$ and its outputs are $[\hat{y}; \hat{x}]$. □

Matlab Hint 5 (`reg`). The command `reg(sys,K,L)` computes a state-space model for a *positive* output-feedback LQG/LQG controller for the process with state-space model `sys` with regulator gain `K` and estimator gain `L`. □

4.8 LTR design example

Example 2 (Aircraft roll-dynamics). Figure 4.4(a) shows Bode plots of the open-loop gain for the state-feedback LQR state-feedback controller vs. the open-loop gain for several output-feedback LQG/LQR controllers The LQR controller was designed using the con-



(a) (b)

Figure 4.4. Bode plots of the open-loop gain and closed-loop step response

trolled output $z := \begin{bmatrix} \theta & \gamma\dot{\theta} \end{bmatrix}'$, $\gamma = .1$ and $\rho = .01$. For the LQG state-estimators we used $\bar{B} = B$ and $R_N = \sigma$ for several values of σ. We can see that, as σ decreases, the range of frequencies over which the open-loop gain of the output-feedback LQG/LQR controller matches that of the state-feedback LQR state-feedback increases. Moreover, at high frequencies the output-feedback controllers exhibit much faster (and better!) decays of the gain's magnitude. □

4.9 Exercises

Exercise 14. Assume that the pair $(-A, \bar{B})$ is stabilizable and that the pair (A, C) is observable. Prove that

Sidebar 30: This result is less interesting than Theorem 6 because often (A, C) is not observable, just detectable. This can happen when we augmented the state of the system to construct a "good" controlled output z but these augmented states are not observable.

(i) There exists a symmetric positive definite solution P to the ARE (4.5), for which $-A - \bar{B}R^{-1}\bar{B}'P$ is Hurwitz. □

(ii) There exists a symmetric positive definite solution $S := P^{-1}$ to the dual ARE (4.7), for which $A - LC$ is Hurwitz.

Solution to Exercise 14. Part (i) is a direct application of Theorem 3 for $N = 0$, using the fact that observability of the pair $(-A, C'QC)$ is equivalent to observability of (A, C) by the eigenvector test.

The fact that $S := P^{-1}$ satisfies (4.7) has already been established from the construction of the dual ARE (4.7). To prove (ii) it remains to show that $A - LC$ is Hurwitz. To this effect we re-write the symmetric of (ii) as

$$(A' - C'QCP^{-1})P + P(A - P^{-1}C'QC) + C'QC + P\bar{B}R^{-1}\bar{B}'P = 0.$$

and therefore

$$(A-LC)'P + P(A-LC) = -S, \qquad\qquad S := C'QC + P\bar{B}R^{-1}\bar{B}'P.$$

The stability of $A-LC$ then follows from the observability Lyapunov test as long as we are able to establish the observability of the pair $(A-LC, S)$.

To show that the pair $(A-LC, S)$ is indeed observable we use the eigenvector test. By contradiction assume that x is an eigenvector of $A-LC$ that lies in the kernel of S, i.e.,

$$\big(A-LC\big)x = \lambda x, \qquad\qquad \big(C'QC + P\bar{B}R^{-1}\bar{B}'P\big)x = 0.$$

These equations imply that $Cx = 0$ and $\bar{B}'Px = 0$ and therefore

$$Ax = \lambda x, \qquad\qquad Cx = 0,$$

which contradicts the fact that the pair (A, C) is observable. □

Chapter 5

LQG/LQR and the Youla Parameterization

Summary

1. Youla parameterization of all stabilizing controllers
2. Q-design of linear controllers
3. Convexity

5.1 Youla Parameterization

Consider a continuous-time LTI system of the form:

$$\dot{x} = Ax + Bu, \qquad y = Cx, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ y \in \mathbb{R}^m, \tag{CLTI}$$

where u is the control signals and y the measured output. We saw in Chapter 3 that an LQG/LQR set-point controller is of the form

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly, \qquad u = -K(\hat{x} - x_{\text{eq}}) + u_{\text{eq}}, \tag{5.1}$$

where $(x_{\text{eq}}, u_{\text{eq}})$ is an equilibrium pair consistent with the desired set-point and $A - LC$, $A - BK$ are Hurwitz.

Suppose however, that instead of (5.1) we use

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly, \qquad u = -K(\hat{x} - x_{\text{eq}}) + u_{\text{eq}} + v, \tag{5.2}$$

where v is the output of an asymptotically stable system driven by the *output estimation error* $\tilde{y} := y - C\hat{x}$:

$$\dot{x}_Q = A_Q x_Q + B_Q \tilde{y}, \qquad v = C_Q x_Q + D_Q \tilde{y}, \qquad \tilde{y} \in \mathbb{R}^m,\ v \in \mathbb{R}^k, \tag{5.3}$$

with A_Q Hurwitz. Defining $\bar{x} := x_{\text{eq}} - \hat{x}$ and using the fact that $Ax_{\text{eq}} + Bu_{\text{eq}} = 0$, we can re-write (5.2) and the output estimation error as

Sidebar 31: When the transfer matrix of (5.3) is equal to zero, we recover the original LQG/LQR set-point controller.

$$\begin{aligned}
\dot{\bar{x}} &= (A - LC - BK)\bar{x} - L(y - Cx_{\text{eq}}) - Bv \\
u &= K\bar{x} + u_{\text{eq}} + v \\
\tilde{y} &= C\bar{x} + (y - Cx_{\text{eq}})
\end{aligned}$$

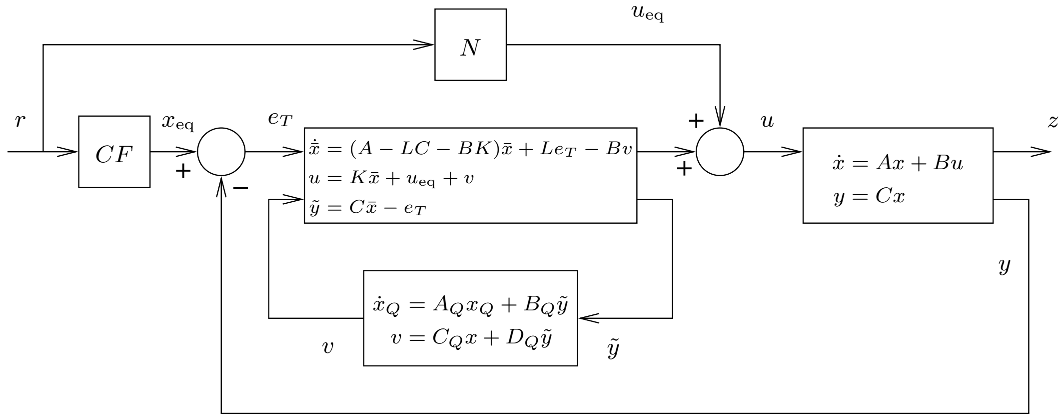


Figure 5.1. LQG/LQR set-point control with Q block.

which corresponds to the control architecture shown in Figure 5.1.

Since $A - LC$ is Hurwitz, the output estimation error $\tilde{y}$ converges to zero and therefore v will also converge to zero because A_Q is Hurwitz. We thus conclude that the controller (5.2)–(5.3) will have the same exact asymptotic behavior as (5.1). In particular, it will also be asymptotically stable. However, *these two controllers will generally result in completely different transient behaviors and different closed-loop transfer functions.*

We just saw that the controller (5.2)–(5.3) has the property that it stabilizes the closed loop *for every LTI asymptotically stable LTI system* (5.3). It turns out that this controller architecture has several other important properties, summarized in the following result:

Theorem 8 (Youla Parameterization).

P3 *The controller* (5.2)–(5.3) *makes the closed-loop asymptotically stable for every Hurwitz matrix* A_Q.

Attention! This realization will generally not be minimal but it is always stabilizable and detectable.

P4 *For every controller transfer function* $\hat{C}(s)$ *that asymptotically stabilizes* (CLTI)*, there exist matrices* A_Q, B_Q, C_Q, D_Q *such that* (5.2)–(5.3) *is a realization for* $\hat{C}(s)$.

Sidebar 32: When noise and disturbances are present, the transfer functions from these signals to any signal in Figure 5.1 can also be expressed as in (5.4).

P5 *The closed-loop transfer function* $\hat{T}(s)$ *from the reference to any signal in Figure 5.1 can always be written as*

$$\hat{T}(s) = \hat{T}_0(s) + \hat{L}(s)\hat{Q}(s)\hat{R}(s), \tag{5.4}$$

where $\hat{Q}(s) := C_Q(sI - A_Q)^{-1}B_Q + D_Q$ *is the transfer function of* (5.3).

Attention! Different inputs and outputs will correspond to different transfer matrices $\hat{T}_0(s)$, $\hat{L}(s)$, and $\hat{R}(s)$ but the closed-loop transfer matrices will always be affine in $\hat{Q}(s)$.

P3 follows from the previous discussion, but P4 and P5 are nontrivial. The proof of these properties can be found in [2, Chapter 5].

Attention! Theorem 8 allows one to construct *every controller that stabilizes an LTI process* and *every stable closed-loop transfer matrix* using a single LQG/LQR controller, by allowing $\hat{Q}(s)$ to range over the space of BIBO stable transfer matrices.

Because of this we say that (5.4) is a *parameterization of the set of all stable closed-loop transfer matrices for the process* (CLTI). □

Exercise 15. Show that the controller (5.2)–(5.3) can also be realized has

$$\begin{bmatrix} \dot{\bar{x}} \\ \dot{x}_Q \end{bmatrix} = \begin{bmatrix} A - LC - BK - BD_QC & -BC_Q \\ B_QC & A_Q \end{bmatrix} \begin{bmatrix} \bar{x} \\ x_Q \end{bmatrix} - \begin{bmatrix} L + BD_Q \\ -B_Q \end{bmatrix} (y - Cx_{\text{eq}})$$

$$u = \begin{bmatrix} K + D_QC & C_Q \end{bmatrix} \begin{bmatrix} \bar{x} \\ x_Q \end{bmatrix} + D_Q(y - Cx_{\text{eq}}) + u_{\text{eq}}. \qquad \square$$

5.2 Q-design

The Youla parameterization is a powerful theoretical result with multiple practical applications. We use it here to improve upon LQG/LQR controllers by numerical optimization. To this effect, suppose that we have designed an LQG/LQR controller for (CLTI) but are not satisfied with some of the closed-loop responses. For concreteness, we assume that we wish to enforce that

1. The step response from the reference $r(t) \in \mathbb{R}$ to a scalar output $s(t) \in \mathbb{R}$ satisfies

$$s_{\min}(t) \le s(t) \le s_{\max}(t), \quad \forall t \in [t_{\min}, t_{\max}]. \tag{5.5}$$

Sidebar 33: To enforce a 1% settling time smaller than or equal to t_{settling} one would chose $z_{\min}(t) = .99$, $z_{\max}(t) = 1.01$, $\forall t \ge t_{\text{settling}}$.

2. The frequency response from the reference $r(t) \in \mathbb{R}$ to a scalar output $w(t)$ satisfies

$$|\hat{h}(j\omega)| \le \ell(\omega), \quad \forall \omega \in [\omega_{\min}, \omega_{\max}], \tag{5.6}$$

where $\hat{h}(s)$ is the transfer function from r to w.

The basic idea is to search for a $k \times m$ BIBO stable transfer matrix $\hat{Q}(s)$, for which the controller in Figure 5.1 satisfies the desired properties. The search for the transfer function will be carried out numerically.

For numerical tractability, one generally starts by selecting N, $k \times m$ BIBO stable transfer matrices

$$\{\hat{Q}_1(s), \hat{Q}_2(s), \cdots, \hat{Q}_N\},$$

and restricts the search to linear combinations of these, i.e., to transfer matrices of the form

$$\hat{Q}(s) := \sum_{i=1}^{N} \alpha_i \hat{Q}_i(s), \tag{5.7}$$

where $\alpha := \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_N \end{bmatrix}$ is an N-parameter vector to be optimized. For this choice of $\hat{Q}(s)$, any closed-loop transfer function can be written as

$$\hat{T}(s) = \hat{T}_0(s) + \sum_{i=1}^{N} \alpha_i \hat{T}_i(s), \qquad \hat{T}_i(s) := \hat{L}(s)\hat{Q}_i(s)\hat{R}(s).$$

Therefore the step response $s(t)$ and the transfer function $\hat{h}(s)$ can also be expressed as

$$s(t) = s_0(t) + \sum_{i=1}^{N} \alpha_i s_i(t), \qquad \hat{h}(s) = \hat{h}_0(s) + \sum_{i=1}^{N} \alpha_i \hat{h}_i(s). \tag{5.8}$$

Sidebar 34: The $s_i(t)$ and $\hat{h}_i(s)$ can be computed using a Simulink diagram as in Figure 5.1.

The requirements (5.5)–(5.6), lead to the following feasibility problem:

Problem 1 (Feasibility). Find a N-vector $\alpha := \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_N \end{bmatrix}$ such that

$$\left| \hat{h}_0(j\omega) + \sum_{i=1}^{N} \alpha_i \hat{h}_i(j\omega) \right| \leq \ell(\omega), \qquad \forall \omega \in [\omega_{\min}, \omega_{\max}],$$

$$s_{\min}(t) \leq s_0(t) + \sum_{i=1}^{N} \alpha_i s_i(t) \leq s_{\max}(t), \qquad \forall t \in [t_{\min}, t_{\max}]. \qquad \square$$

From a numerical perspective, it is often preferable to work with an optimization problem:

Matlab hint 6: `fmincon` can be used to solve nonlinear constrained optimizations of this type. However, this function does not fully explore convexity.

Problem 2 (Minimization). Find an N-vector $\alpha := \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_N \end{bmatrix}$ that

$$\text{minimizes} \qquad J := \max_{\omega \in [\omega_{\min}, \omega_{\max}]} \frac{1}{\ell(\omega)} \left| \hat{h}_0(j\omega) + \sum_{i=1}^{N} \alpha_i \hat{h}_i(j\omega) \right|,$$

$$\text{subject to} \qquad s_{\min}(t) \leq s_0(t) + \sum_{i=1}^{N} \alpha_i s_i(t) \leq s_{\max}(t), \quad \forall t \in [t_{\min}, t_{\max}].$$

Problem 1 has a feasible solution if and only if the minimum J in Problem 2 is smaller than or equal to one. $\square$

Sidebar 34 (Computing the $s_i(t)$ and $\hat{h}_i(s)$). The step response $s_0(t)$ and the transfer function $h_0(s)$ correspond to the original LQG/LQR controller ($\hat{Q}(s) = 0$).

Matlab hint 7: When using MATLAB/Simulink to compute these transfer functions, one generally does not obtain minimal realization, so one may want to use `minreal(sys)` to remove unnecessary states.

To compute $s_i(t)$ and $h_i(s)$, one sets (5.3) to be a realization of $\hat{Q}_i(s)$ and computes the closed-loop step response and transfer function. From (5.8), we know that these must be equal to

$$s_0(t) + s_i(t), \qquad \hat{h}_0(s) + \hat{h}_i(s),$$

from which we can recover $s_i(t)$ and $\hat{h}_i(s)$ by subtracting $s_0(t)$ and $\hat{h}_0(s)$, respectively. $\square$

Sidebar (Selecting the $\hat{Q}_i(s)$). There is no systematic procedure to select the $\hat{Q}_i(s)$. Often one decides on a few positions for the poles

$$\{p_1, p_2, \dots, p_q\},$$

and then selects the $\hat{Q}_i(s)$ to be $k \times m$ transfer matrices with all entries but one equal to zero. The nonzero entries will be of the form

$$\frac{s^i}{\prod_{j=1}(s - p_j)}, \quad i \in \{1, \dots, q\},$$

which leads to $N := k \times m \times q$ distinct transfer matrices. To restrict the $\hat{Q}_i(s)$ to be strictly proper one would only use $i < q$. $\square$

5.3 Convexity

One needs to be quite careful in choosing which criteria to optimize.

A closed-loop *control specification is said to be convex* if given any two closed-loop transfer functions $\hat{T}_1(s)$ and $\hat{T}_2(s)$ that satisfy the constraint, the closed-loop transfer function

$$\frac{\hat{T}_1(s) + \hat{T}_2(s)}{2}$$

also satisfies the constraint. A closed-loop *optimization criteria is said to be convex* if given any two closed-loop transfer functions $\hat{T}_1(s)$ and $\hat{T}_2(s)$ for which the value of the criteria is J_1 and J_2, respectively, the closed-loop transfer function

$$\frac{\hat{T}_1(s) + \hat{T}_2(s)}{2}$$

leads to a value for the criteria smaller than or equal to

$$\frac{J_1 + J_2}{2}.$$

Convex control specifications and criteria are especially appealing from a numerical perspective because local minima are always global minima. However, criteria that are convex but not differentiable can sometimes trap optimization algorithms that do not fully explore convexity.

Convexity holds for criteria of the type

$$J := \max_{t\in[t_{\min},t_{\max}]} f(s(t), t), \qquad J := \int_{t_{\min}}^{t_{\max}} f(s(t), t)dt$$

$$J := \max_{\omega\in[\omega_{\min},\omega_{\max}]} f\big(\hat{h}(j\omega), \omega\big), \qquad J := \int_{\omega_{\min}}^{\omega_{\max}]} f\big(\hat{h}(j\omega), \omega\big)d\omega,$$

or control specifications of the type

$$f(s(t), t) \le 0, \ \ \forall t \in [t_{\min}, t_{\max}], \qquad \int_{t_{\min}}^{t_{\max}} f(s(t), t)dt \le 0$$

$$f\big(\hat{h}(j\omega), \omega\big), \ \ \forall \omega \in [\omega_{\min}, \omega_{\max}], \qquad \int_{\omega_{\min}}^{\omega_{\max}]} f\big(\hat{h}(j\omega), \omega\big)d\omega,$$

where $s(t)$ is a closed-loop "time-response," $\hat{h}(s)$ a closed-loop "frequency response," and $f(\alpha, \beta)$ is a convex function on α, for every β. However, the criteria on the left-hand-side are generally not differentiable. This is the case of the criterion in Problem 2.

The settling time

$$J := \min\{T \ge 0 : |s(t) - 1| \le 1\%, \forall t \ge T\},$$

is not convex and therefore one should avoid using it directly as an optimization criteria. Instead, one should solve a sequence of problems in which one successfully enforces decreasing upper-bounds on the settling time, until the problem becomes unfeasible. The convexity properties of controller specifications is extensively discussed in [1, Chapter 8].

Bibliography

[1] S. P. Boyd and C. H. Barratt. *Linear Controller Design: Limits of Performance.* Prentice-Hall, New Jersey, 1991.

[2] G. E. Dullerud and F. Paganini. *A Course in Robust Control Theory.* Number 36 in Texts in Applied Mathematics. Springer, New York, 1999.

[3] G. F. Franklin, J. D. Powell, and A. Emami-Naeini. *Feedback Control of Dynamic Systems.* Prentice Hall, Upper Saddle River, NJ, 4th edition, 2002.

[4] H. Kwakernaak and R. Sivan. *Linear optimal control systems.* Wiley Interscience, New York, 1972.

[5] R. S. Sánchez-Peña and M. Sznaier. *Robust Systems: Theory and Applications.* Adaptive and Learning Systems for Signal Processing Communications, and Control. John Wiley & Sons, Inc., New York, 1998.

[6] J. V. Vegte. *Feedback Control Systems.* Prentice Hall, New Jersey, 3rd edition, 1994.

Index