ELEC4410

Control Systems Design

Lecture 14: Controllability

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Outline

- Controllability Definition and Tests
- Controllability and Algebraic Equivalence
- Controllability Gramian
- Controllability after Sampling
- Examples

Controllability

In the next few classes we will discuss two fundamental concepts of system theory: those of **controllability** and **observability**.

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- **Controllability** is the property that indicates if the behaviour of a system can be controlled by acting on its inputs.
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We start by considering **controllability** of continuous-time systems.

Controllability

Consider the LTI system represented by the n-states, q-inputs state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{SE}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times q}$.

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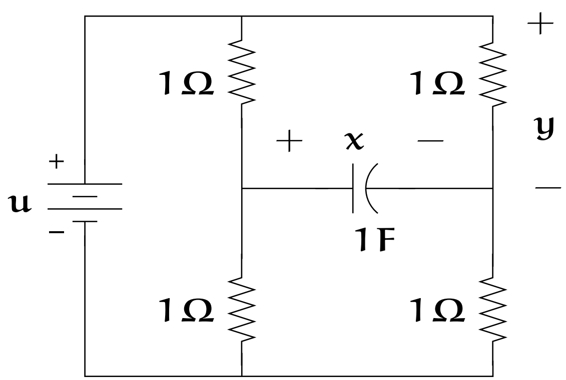
Controllability: The state equation (SE) or the pair (A, B) is said to be controllable if for any initial state $x(0) = x_0$ and any final state x_1, there exists an input that transfer x_0 to x_1 in a finite time. Otherwise, (SE) or (A, B) is said to be uncontrollable.

This definition requires only that the input be capable of driving the state anywhere in the state space space in a finite time; what trajectory the state takes is not specified.

Controllability

Example (Uncontrollable systems). Consider the electric system on the left in the figure below. It is a system of first order; state variable x: voltage on the capacitor.

If the capacitor has no initial charge, $x(0) = 0$, then $x(t) = 0$ for all $t \geq 0$, no matter what input is applied. The input has no effect over the voltage across the capacitor. This system, or more precisely, a state equation that describes it, is not controllable.



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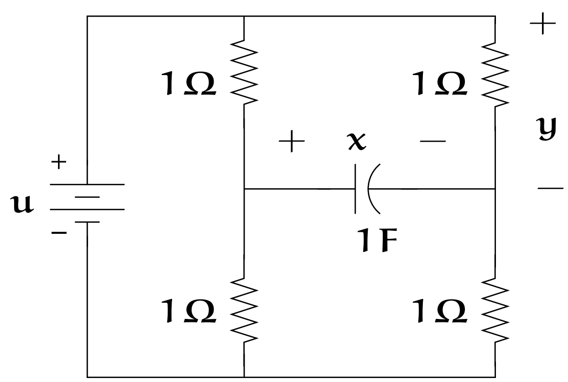
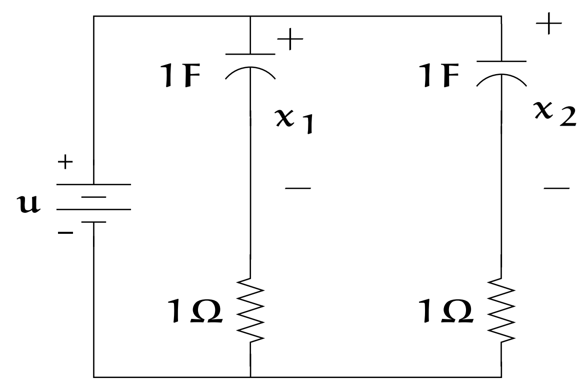
The system on the right has two state variables. The input can transfer x_1 *or* x_2 to any desired value, but no matter what input is applied, $x_1(t)$ will always equal $x_2(t)$. This system is not controllable either.





Controllability Tests

Theorem (Controllability Tests). The following statements are equivalent.

1. The n-dimensional pair (A, B) is controllable.
2. The **Controllability Matrix**

   $$\mathcal{C} \triangleq \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

   has rank n (full row rank).
3. The $n \times n$ matrix

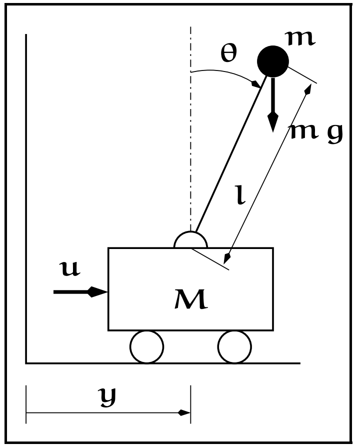
   $$W_c(t) = \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau$$

   is nonsingular for all $t > 0$.

Controllability Tests

Example. The linearised state space equation of an inverted pendulum system is given by

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u$$



We compute the controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -10 \\ -2 & 0 & -10 & 0 \end{bmatrix}$$

which has rank 4 (i.e., it is full rank) $\Rightarrow$ the system is **controllable**.

If θ were slightly different from 0, we know then that there exists a control u that will return it to the equilibrium in finite time. $\square$

Controllability & Algebraic Equivalence

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Indeed, consider the pair $(\mathbf{A}, \mathbf{B})$ with controllability matrix

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

and an algebraic equivalent pair $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$, where $\bar{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$ and $\bar{\mathbf{B}} = \mathbf{P}\mathbf{B}$, and $\mathbf{P}$ is a nonsingular matrix. Then the controllability matrix of the pair $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ is

$$\begin{aligned} \bar{\mathcal{C}} &= \begin{bmatrix} \bar{\mathbf{B}} & \bar{\mathbf{A}}\bar{\mathbf{B}} & \cdots & \bar{\mathbf{A}}^{n-1}\bar{\mathbf{B}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{P}\mathbf{B} & \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\mathbf{P}\mathbf{B} & \cdots & \mathbf{P}\mathbf{A}^{n-1}\mathbf{P}^{-1}\mathbf{P}\mathbf{B} \end{bmatrix} \\ &= \mathbf{P} \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} = \mathbf{P}\mathcal{C}. \end{aligned}$$

Because $\mathbf{P}$ is nonsingular, **rank** $\mathcal{C}$ = **rank** $\bar{\mathcal{C}}$

Controllability Gramian

Controllability Gramian

The matrix $W_c(t)$ introduced to check controllability of (A, B) can be used to construct an **open loop** control signal $u(t)$ that will take the state x from any x_0 to any x_1 in finite time.

$$\boxed{W_c(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau}$$

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Such a control law is given by

$$u(t) = -B^T e^{A^T(t_1 - t)} W_c^{-1}(t_1)(e^{At_1}x_0 - x_1).$$

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This control law uses the least amount of energy to transfer x from x_0 to x_1 in time t_1. This means that for any other control $\tilde{u}(t)$ performing the same transfer,

$$\int_0^{t_1} \|\tilde{u}(\tau)\|^2 d\tau \geq \int_0^{t_1} \|u(\tau)\|^2 d\tau.$$

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For example, if $x_0 = 0$, the minimum control energy is

$$\int_0^{t_1} \|u(\tau)\|^2 d\tau = -x_1^T W_c^{-1}(t_1) \left(\int_0^{t_1} e^{A(t_1-\tau)} B B^T e^{A^T(t_1-\tau)} d\tau \right) W_c^{-1}(t_1)(-x_1)$$

$$= x_1^T W_c^{-1}(t_1) x_1 = \|W_c^{-\frac{1}{2}}(t_1) x_1\|^2.$$

Controllability Gramian

If the matrix A is **Hurwitz** (all eigenvalues with negative real part), then $W_c(t)$ converges for $t \to \infty$, and then we denote it simply by W_c

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If we desire to drive the state x from 0 to x_1 in infinite time, $t_1 \to \infty$, we find that the least required control energy would be

$$\int_0^\infty \|u(\tau)\|^2 d\tau = \|W_c^{-\frac{1}{2}} x_1\|^2.$$

Note that the closer to zero any eigenvalue of W_c is, the closer to singular would W_c be, and the larger would be the minimum energy required to drive the state to x_1.

Controllability Gramian

Note that we do not need to slove an infinite integral to compute $\mathbf{W_c}$. If $(\mathbf{A}, \mathbf{B})$ is controllable ($\mathcal{C}$ has full row rank), $\mathbf{W_c}$ is the unique solution of the linear **Lyapunov matrix equation**

$$\mathbf{A}\mathbf{W_c} + \mathbf{W_c}\mathbf{A}^\mathsf{T} = -\mathbf{B}\mathbf{B}^\mathsf{T},$$

which can be solved with MATLAB with `Wc = lyap(A,B*B')`, or by using the function `Wc = gram(SYS,'c')`.

Controllability after Sampling

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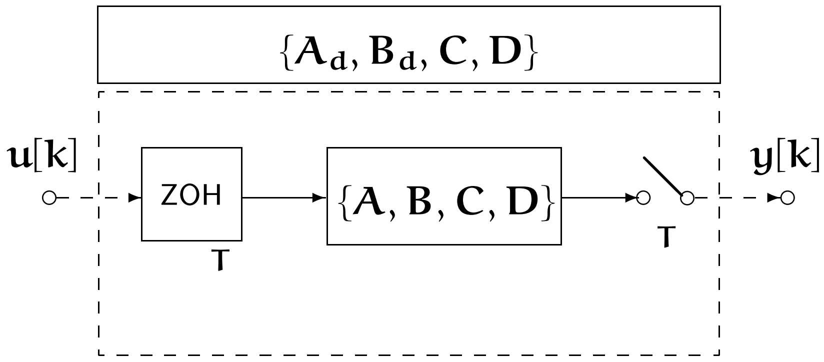
Most control systems are implemented digitally, for which we need a **discrete-time** model of the system.



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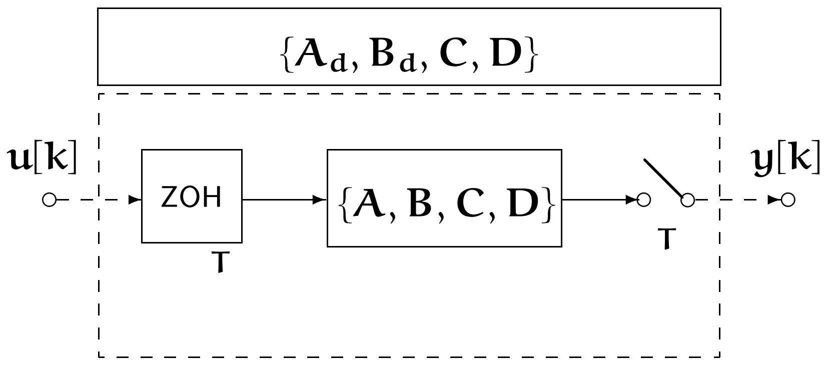

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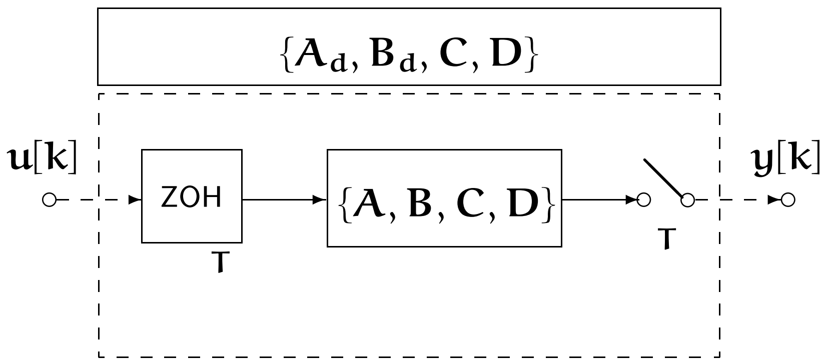

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$$\boxed{\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}} \quad \Rightarrow \quad \boxed{\begin{aligned} x[k+1] &= A_d\,x[k] + B_d\,u[k] \\ y[k] &= C\,x[k] + D\,u[k] \end{aligned}}$$

where $A_d = e^{AT}$ and $B_d = \int_0^T e^{A\tau}\,B\,d\tau$.

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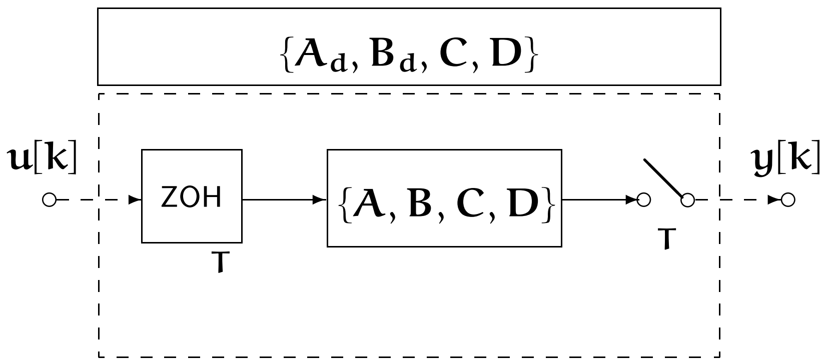

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where $A_d = e^{AT}$ and $B_d = \int_0^T e^{A\tau}\,B\,d\tau$.

If the continuous-time system is controllable, would the discretised system be always controllable?

Controllability after Sampling

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Theorem (Nonpathological Sampling). If the pair $[A, B]$ is controllable, then the discretised pair $[A_d, B_d]$ is controllable with sampling time T if for any two eigenvalues λ_i, λ_j of A such that **Re**$[\lambda_i - \lambda_j] = 0$, the **nonpathological sampling condition**

$$\mathbf{Im}[\lambda_i - \lambda_j] \neq \frac{2\pi m}{T}, \qquad \text{for } m = 1, 2, \ldots$$

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The Theorem gives a **sufficient** condition that preserves controllability after sampling. This condition is also **necessary** for systems with a single input.

Controllability after Sampling

Example (Pathological Sampling). Consider the controllable continuous-time system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

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Its exact discretisation with sampling period T is

$$x[k+1] = \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix} x[k] + \begin{bmatrix} 1 - \cos T \\ \sin T \end{bmatrix} u[k].$$

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Note that if $T = m\pi$, with $m = 1, 2, \ldots$, this system becomes uncontrollable, i.e.,

$$x[k+1] = \begin{bmatrix} (-1)^m & 0 \\ 0 & (-1)^m \end{bmatrix} x[k] + \begin{bmatrix} 1 - (-1)^m \\ 0 \end{bmatrix} u[k].$$

Controllability after Sampling

A couple of final remarks concerning controllability and sampling:

- The nonpathological sampling condition only applies to systems with complex eigenvalues; a discretised system with only real eigenvalues is controllable for all $T > 0$ if its continuous-time counterpart is.

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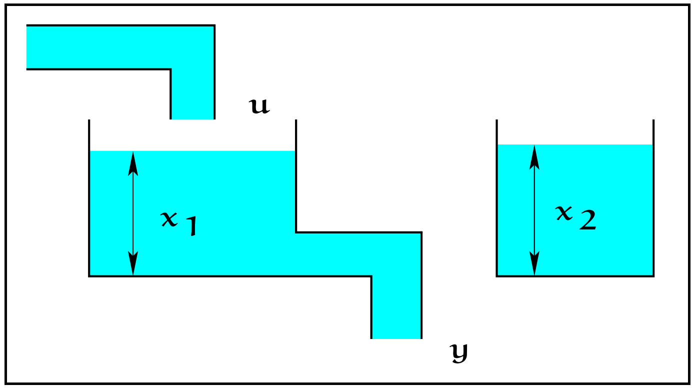
- The nonpathological sampling condition only applies to systems with complex eigenvalues; a discretised system with only real eigenvalues is controllable for all $T > 0$ if its continuous-time counterpart is.
- The nonpathological sampling condition is only **sufficient** for a MIMO system; if sampling is pathological, controllability **may** be lost after sampling.

Controllability Examples

Controllability Examples

Example.



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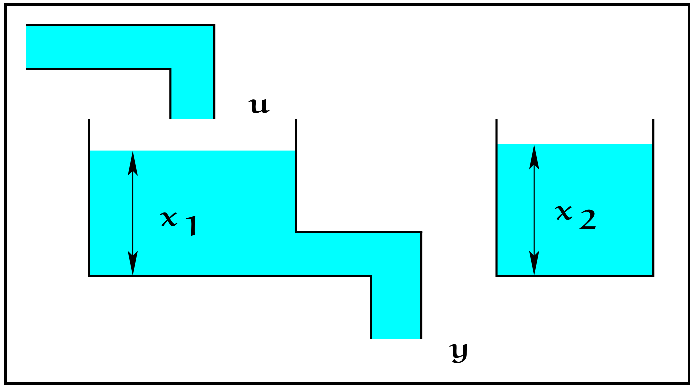

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A linearised model of this system with unitary parameters gives

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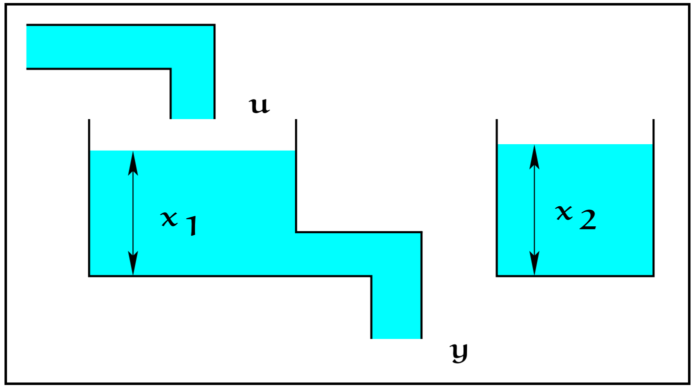

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The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

which is not full rank, so the system is not controllable.

Controllability Examples

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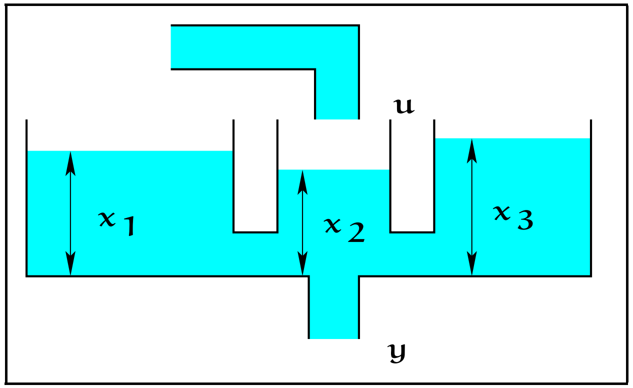

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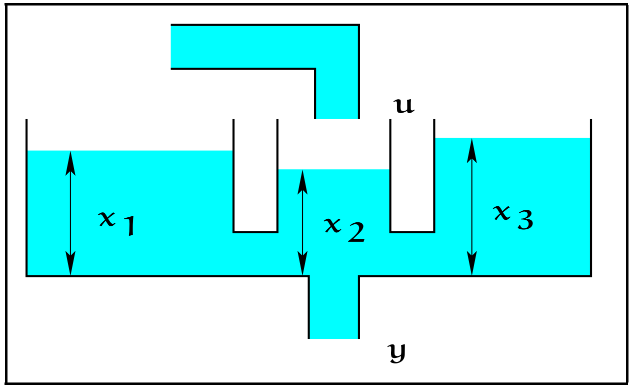

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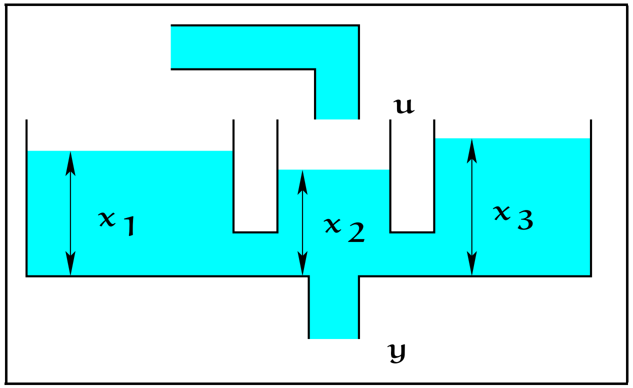

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The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -4 \\ 1 & -3 & 11 \\ 0 & 1 & -4 \end{bmatrix}$$

which has rank 2, showing that the system is not controllable.

Controllability Examples

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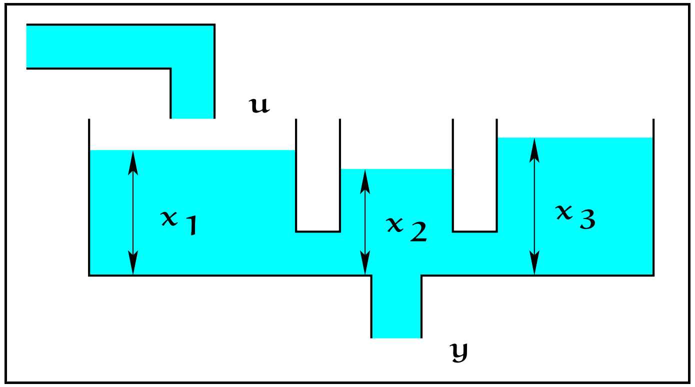

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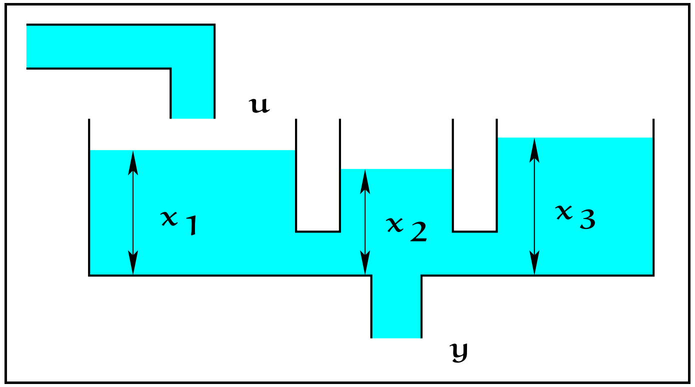

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$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}(t)$$
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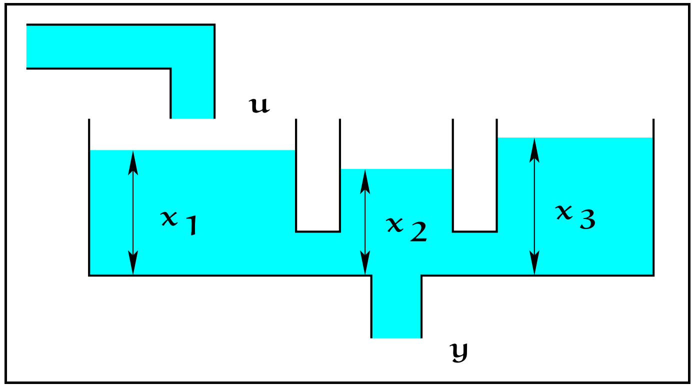

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The controllability matrix is now

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

which has rank 3, showing that the system is controllable.

Summary

- **Controllability** is a fundamental property of a system which determines if it is possible to drive its state $\mathbf{x}$ to any desired position by an appropriate choice of the input signal $\mathbf{u}(t)$.

Summary

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