

ELEC4410
Control Systems Design
Lecture 17: State Feedback

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Outline

- ▶ Overview of Control Design via State Space Methods
- ▶ State Feedback Design
- ▶ State Feedback Stabilisation

Overview of Control via State Space

The **linear systems theory** that we've been discussing is the basis for **linear control design theory** in state space, which we will discuss from this lecture on.

Linear state space control theory involves modifying the behaviour of an m -input, p -output, n -state system

$$\begin{aligned}\dot{\mathbf{x}}(\mathbf{t}) &= \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}\mathbf{u}(\mathbf{t}) \\ \mathbf{y}(\mathbf{t}) &= \mathbf{C}\mathbf{x}(\mathbf{t}),\end{aligned}\tag{OL}$$

which we call **the plant**, or **open loop state equation**, by application of a control law of the form

$$\mathbf{u}(\mathbf{t}) = \mathbf{N}\mathbf{r}(\mathbf{t}) - \mathbf{K}\mathbf{x}(\mathbf{t}),\tag{U}$$

in which $\mathbf{r}(\mathbf{t})$ is the new (reference) input signal. The matrix \mathbf{K} is the **state feedback gain** and \mathbf{N} the **feedforward gain**.

Overview of Control via State Space

Substitution of (U) into (OL) gives the **closed-loop state equation**

$$\begin{aligned}\dot{\mathbf{x}}(\mathbf{t}) &= (\mathbf{A} - \mathbf{BK})\mathbf{x}(\mathbf{t}) + \mathbf{BNr}(\mathbf{t}) \\ \mathbf{y}(\mathbf{t}) &= \mathbf{Cx}(\mathbf{t}).\end{aligned}\tag{CL}$$

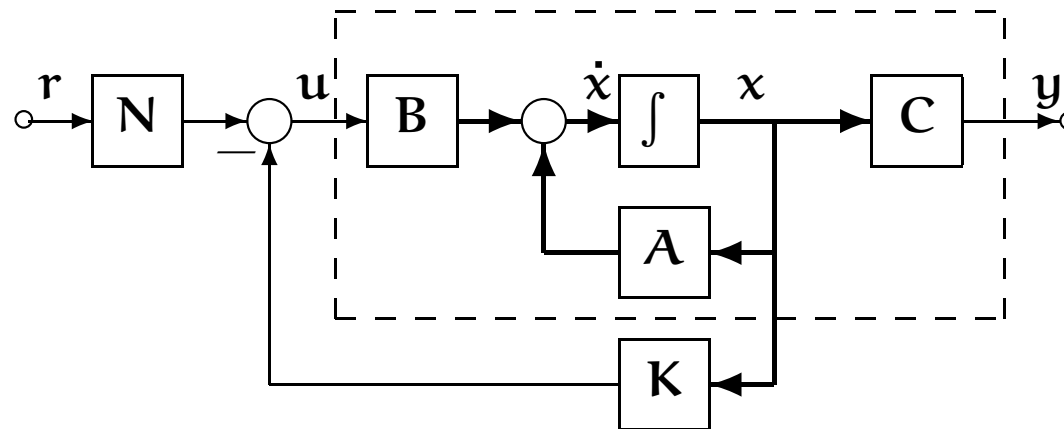
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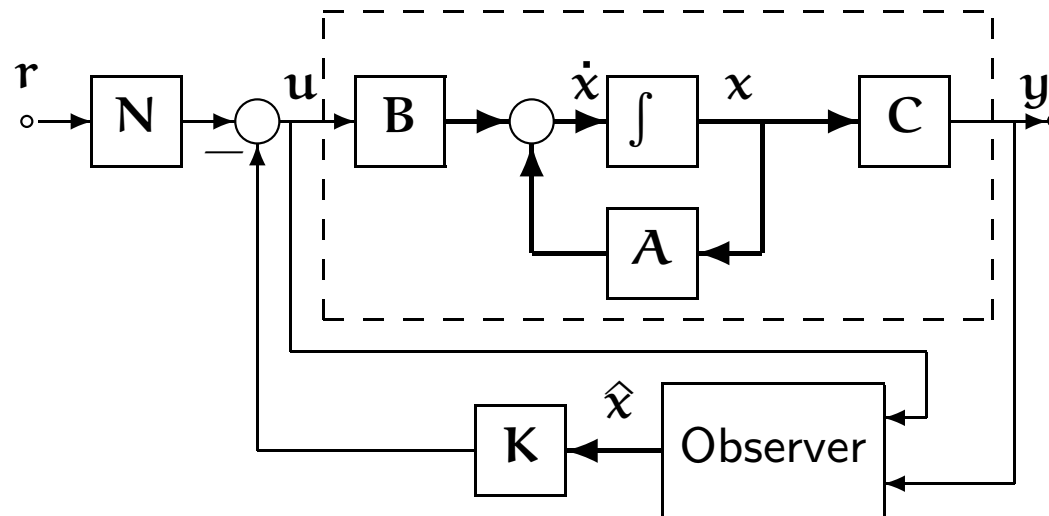


State feedback with feedforward precompensation

This type of control is said to be **static**, because u only depends on the present values of the state x and the reference r . Note that *it requires that all states of the system be measured*.

Overview of Control via State Space

When not all the states of the system are measurable, we resource to their *estimation* by means of an *observer*, or *state estimator*, which reconstructs the state from measurements of the inputs $\mathbf{u}(t)$ and outputs $\mathbf{y}(t)$.



Output feedback by estimated state feedback

The combination of state feedback and state estimation yields a **dynamic** output feedback controller.

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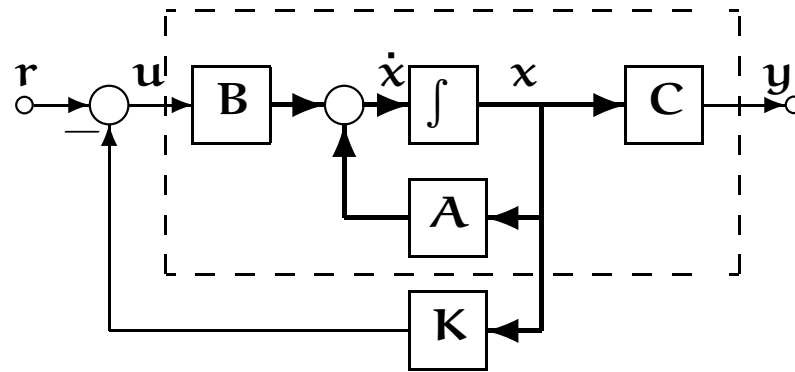
Our basic aim in this part of the course is

To learn how to design a linear control system by dynamic output feedback (state feedback + observer) to satisfy the desired closed-loop system specifications in stability and performance.

We start with SISO systems, and then move on to MIMO. Over the end of the course, we will discuss some optimal designs.

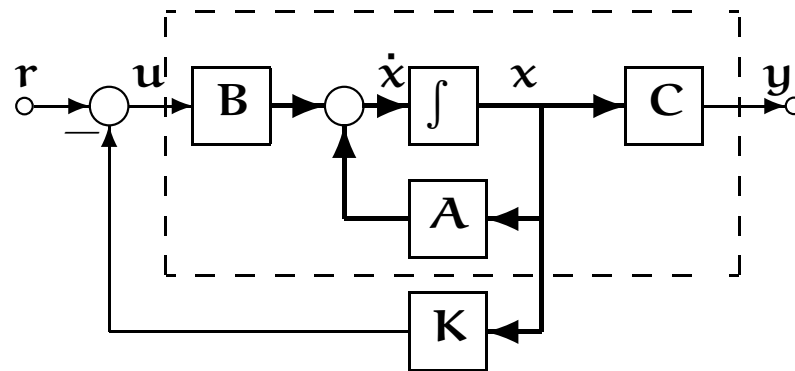
State Feedback Design

We start by considering SISO systems, and the state feedback control scheme with feedforward precompensation. For simplicity, let's assume for the moment that $\mathbf{N} = \mathbf{1}$.



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An essential system property in state feedback is **Controllability**, and our first important observation is that **Controllability is invariant with respect to state feedback**.

Theorem (Invariance of Controllability with State Feedback). For any $\mathbf{K} \in \mathbb{R}^{1 \times n}$, the pair $(\mathbf{A} - \mathbf{BK}, \mathbf{B})$ is controllable if and only if the pair (\mathbf{A}, \mathbf{B}) is controllable. \square

State Feedback Design

It is interesting to note that, on the other hand, **Observability is not invariant with respect to feedback.**

Example (Loss of Observability after feedback). The system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$
$$\mathbf{y}(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \mathbf{x}(t)$$

is controllable and observable, since its controllability and observability matrices

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad \mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 7 & 4 \end{bmatrix}$$

are nonsingular.

State Feedback Design

Example (Continuation). The state feedback control

$$\mathbf{u}(\mathbf{t}) = \mathbf{r}(\mathbf{t}) - \begin{bmatrix} 3 & 1 \end{bmatrix} \mathbf{x}(\mathbf{t})$$

yields the closed loop state equations

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The controllability matrix for the closed loop state equation is $\mathbf{C}_K = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, which, as expected, is nonsingular, and verifies that the closed-loop system is controllable.

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Observability is *not* invariant with respect to feedback.

State Feedback Design

The following example illustrates what can we achieve with state feedback.

Example (Eigenvalue assignment by state feedback). The plant

$$\dot{\mathbf{x}}(\mathbf{t}) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \mathbf{x}(\mathbf{t}) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}(\mathbf{t})$$

has a matrix \mathbf{A} with the characteristic polynomial

$$\Delta(s) = (s - 1)^2 - 9 = s^2 - 2s - 8 = (s - 4)(s + 2),$$

i.e., eigenvalues at $s = 4$ and $s = -2$. The system is unstable.

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The state feedback $\mathbf{u} = \mathbf{r} - [\mathbf{k}_1 \ \mathbf{k}_2]\mathbf{x}$ gives the closed-loop system

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{B}\mathbf{r} \\ &= \left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} \right) \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{r} \\ &= \begin{bmatrix} 1-k_1 & 3-k_2 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{r}. \end{aligned}$$

State Feedback Design

Example (Continuation). The “new” evolution matrix $\mathbf{A}_K = \mathbf{A} - \mathbf{BK}$ has characteristic polynomial

$$\begin{aligned}\Delta_K(s) &= (s - 1 + k_1)(s - 1) - 3(3 - k_2) \\ &= s^2 + (k_1 - 2)s + (3k_2 - k_1 - 8).\end{aligned}$$

Clearly, the roots of $\Delta_K(s)$, or equivalently, the eigenvalues of the closed-loop system can be arbitrarily assigned by a suitable choice of k_1 and k_2 .

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Clearly, the roots of $\Delta_K(s)$, or equivalently, the eigenvalues of the closed-loop system can be arbitrarily assigned by a suitable choice of k_1 and k_2 .

For instance, for both eigenvalues to be placed at $-1 \pm j2$, the desired characteristic polynomial is $(s + 1 - j2)(s + 1 + j2) = s^2 + 2s + 5$. By equating $k_1 - 1 = 2$ and $3k_2 - k_1 - 8 = 5$ we get $k_1 = 4$ and $k_2 = 17/3$.

Thus, the feedback gain

$$\mathbf{K} = [4 \quad 17/3]$$

will shift the eigenvalues of the system from $4, -2$ to $-1 \pm j2$. □

State Feedback Design

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state feedback allows us to place the eigenvalues of the closed-loop system at any position, and that the state feedback gain can be computed by direct substitution.

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state feedback allows us to place the eigenvalues of the closed-loop system at any position, and that the state feedback gain can be computed by direct substitution.

However,

- ▶ The method of the example is not practical for systems of higher dimensions.
- ▶ It's not clear what role did controllability play in this eigenvalue assignment.

State Feedback Design

To formulate a general result we need to use the **Controller Canonical Form** (CCF) discussed in Lecture 12. Recall that if $\mathcal{C} = [\mathbf{B} \ \mathbf{A}\mathbf{B} \ \dots \ \mathbf{A}^{n-1}\mathbf{B}]$ is full row rank, then we can always make a change of coordinates under which the state matrices have the form

$$\bar{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad \bar{\mathbf{B}} = \mathbf{P}\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$\bar{\mathbf{C}} = \mathbf{C}\mathbf{P}^{-1} = [\beta_1 \ \beta_2 \ \dots \ \beta_{n-1} \ \beta_n].$$

These matrices arise from the change of coordinates $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ where

$$\mathbf{P}^{-1} = \mathcal{C}\bar{\mathcal{C}}^{-1} \quad \text{with} \quad \mathcal{C} = [\mathbf{B} \ \mathbf{A}\mathbf{B} \ \dots \ \mathbf{A}^{n-1}\mathbf{B}]$$
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State Feedback Design

Theorem (Eigenvalue assignment by state feedback). if the state equation

$$\begin{aligned}\dot{\mathbf{x}}(\mathbf{t}) &= \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}\mathbf{u}(\mathbf{t}) \\ \mathbf{y}(\mathbf{t}) &= \mathbf{C}\mathbf{x}(\mathbf{t}),\end{aligned}$$

is controllable, then the state feedback control law

$$\boxed{\mathbf{u} = \mathbf{r} - \mathbf{K}\mathbf{x}}, \quad \text{where } \mathbf{K} \in \mathbb{R}^{1 \times n},$$

assigns the eigenvalues of the closed-loop state equation

$$\begin{aligned}\dot{\mathbf{x}}(\mathbf{t}) &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(\mathbf{t}) + \mathbf{B}\mathbf{r}(\mathbf{t}) \\ \mathbf{y}(\mathbf{t}) &= \mathbf{C}\mathbf{x}(\mathbf{t}),\end{aligned}$$

to any desired, arbitrary locations, provided that complex eigenvalues are assigned in conjugate pairs.

State Feedback Design

Proof: If the system is controllable, we can take it to its CCF by the change of coordinates $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$, which yields $\bar{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ and $\bar{\mathbf{B}} = \mathbf{B}\mathbf{P}$. It is not difficult to verify that

$$\bar{\mathbf{c}} \triangleq [\bar{\mathbf{B}}, \bar{\mathbf{A}}\bar{\mathbf{B}}, \dots, \bar{\mathbf{A}}^{n-1}\bar{\mathbf{B}}] = \mathbf{P}[\mathbf{B}, \mathbf{A}\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}] = \mathbf{P}\mathbf{c},$$

and thus $\mathbf{P}^{-1} = \mathbf{c}\bar{\mathbf{c}}^{-1}$.

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On substituting $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ in the state feedback law

$$\mathbf{u} = \mathbf{r} - \mathbf{K}\mathbf{x} = \mathbf{r} - \mathbf{K}\mathbf{P}^{-1}\bar{\mathbf{x}} \triangleq \mathbf{r} - \bar{\mathbf{K}}\bar{\mathbf{x}},$$

Since $\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}} = \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{P}^{-1}$, we see that $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}}$ are similar, and thus **have the same eigenvalues**.

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Proof: If the system is controllable, we can take it to its CCF by the change of coordinates $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$, which yields $\bar{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ and $\bar{\mathbf{B}} = \mathbf{B}\mathbf{P}$. It is not difficult to verify that

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Now, say that $\lambda_1, \lambda_2, \dots, \lambda_n$ are the desired closed-loop eigenvalue locations. We can then generate the **desired characteristic polynomial**

$$\begin{aligned}\Delta_{\mathbf{K}}(s) &= (s - \lambda_1)(s - \lambda_2) \dots (s - \lambda_n) \\ &= s^n + \bar{\alpha}_1 s^{n-1} + \dots + \bar{\alpha}_n.\end{aligned}$$

State Feedback Design

If we choose

$$\bar{\mathbf{K}} = [(\bar{\alpha}_1 - \alpha_1), (\bar{\alpha}_2 - \alpha_2), \dots, (\bar{\alpha}_n - \alpha_n)],$$

the closed-loop state equation becomes (in the $\bar{\mathbf{x}}$ coordinates)

$$\begin{aligned} \dot{\bar{\mathbf{x}}}(t) &= (\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}})\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}\mathbf{r}(t) \\ &= \left(\begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} - \begin{bmatrix} (\bar{\alpha}_1 - \alpha_1) & (\bar{\alpha}_2 - \alpha_2) & \cdots & (\bar{\alpha}_n - \alpha_n) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right) \bar{\mathbf{x}}(t) + \bar{\mathbf{B}}\mathbf{r}(t) \\ &= \begin{bmatrix} -\bar{\alpha}_1 & -\bar{\alpha}_2 & \cdots & -\bar{\alpha}_{n-1} & -\bar{\alpha}_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \bar{\mathbf{x}}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mathbf{r}(t). \end{aligned}$$

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Because the **closed-loop** evolution matrix $(\bar{\mathbf{A}} - \bar{\mathbf{B}}\bar{\mathbf{K}})$ is still in companion form, we see from the last expression that its characteristic polynomial is the desired one $\Delta_{\mathbf{K}}(s)$. Finally, from $\bar{\mathbf{K}} = \mathbf{K}\mathbf{P}^{-1}$, we get that $\mathbf{K} = \bar{\mathbf{K}}\mathbf{P}$. \square

State Feedback Design

The matrix $\bar{\mathbf{C}}$ can be easily build from the coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ of the characteristic polynomial of \mathbf{A} or $\bar{\mathbf{A}}$ as

$$\bar{\mathbf{c}} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \dots & \alpha_{n-2} & \alpha_{n-1} \\ 0 & 1 & \alpha_1 & \dots & \alpha_{n-3} & \alpha_{n-2} \\ 0 & 0 & 1 & \dots & \alpha_{n-4} & \alpha_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \alpha_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}^{-1} \quad (\text{note the inverse!})$$

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In closed-loop, once the eigenvalue assignment is performed, the system transfer function from \mathbf{r} to \mathbf{y} is given by

$$\mathbf{G}(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \dots + \beta_n}{s^n + \bar{\alpha}_1 s^{n-1} + \bar{\alpha}_2 s^{n-2} + \dots + \alpha_n}.$$

The transfer function **has poles at the new, desired locations**. However, **the zeros of the system are the same as in the open-loop system**.

State feedback can arbitrarily assign the system poles (eigenvalues), but has **no effect on the system zeros**.

State Feedback Design

Procedure for pole placement by state feedback (Bass-Gura Formula)

1. Obtain the coefficients of the open loop characteristic polynomial $\Delta(s) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$.
2. Form the controllability matrices $\mathcal{C} = [B \ AB \ \dots \ A^{n-1}B]$ and

$$\bar{\mathcal{C}} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \dots & \alpha_{n-2} & \alpha_{n-1} \\ 0 & 1 & \alpha_1 & \dots & \alpha_{n-3} & \alpha_{n-2} \\ 0 & 0 & 1 & \dots & \alpha_{n-4} & \alpha_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \alpha_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}^{-1} \quad (\text{note the inverse!})$$

3. Select the coefficients of the desired **closed-loop** characteristic polynomial $\Delta_K(s) = s^n + \bar{\alpha}_1 s^{n-1} + \dots + \bar{\alpha}_n$ and build the state-feedback gain in \bar{x} coordinates,

$$\bar{\mathbf{K}} = [(\bar{\alpha}_1 - \alpha_1) \ (\bar{\alpha}_2 - \alpha_2) \ \dots \ (\bar{\alpha}_n - \alpha_n)]$$

4. Compute the state-feedback gain in the original x coordinates

$$\mathbf{K} = \bar{\mathbf{K}}\bar{\mathcal{C}}^{-1}$$

State Feedback Stabilisation

We have seen that if a state equation is controllable, then we can assign its eigenvalues arbitrarily by state feedback. But, what happens when the state equation is **not** controllable?

State Feedback Stabilisation

We have seen that if a state equation is controllable, then we can assign its eigenvalues arbitrarily by state feedback. But, **what happens when the state equation is not controllable?**

We know that we can take any state equation to the **Controllable/Uncontrollable Canonical Form**

$$\begin{bmatrix} \dot{\bar{x}}_e \\ \dot{\bar{x}}_{\bar{e}} \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{\mathbf{A}}_e & \bar{\mathbf{A}}_{12} \\ 0 & \bar{\mathbf{A}}_{\bar{e}} \end{bmatrix}}_{\bar{\mathbf{A}}} \begin{bmatrix} \bar{x}_e \\ \bar{x}_{\bar{e}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_e \\ 0 \end{bmatrix} u$$

Because the evolution matrix $\bar{\mathbf{A}}$ is **block-triangular**, its eigenvalues are the union of the eigenvalues of the diagonal blocks: $\bar{\mathbf{A}}_e$ and $\bar{\mathbf{A}}_{\bar{e}}$.

State Feedback Stabilisation

The state feedback law

$$\begin{aligned}\mathbf{u} &= \mathbf{r} - \mathbf{K}\mathbf{x} \\ &= \mathbf{r} - \bar{\mathbf{K}}\bar{\mathbf{x}} \\ &= \mathbf{r} - [\bar{\mathbf{K}}_e \quad \bar{\mathbf{K}}_{\tilde{e}}] \begin{bmatrix} \dot{\bar{\mathbf{x}}}_e \\ \dot{\bar{\mathbf{x}}}_{\tilde{e}} \end{bmatrix}\end{aligned}$$

yields the closed-loop system

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_e \\ \dot{\bar{\mathbf{x}}}_{\tilde{e}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_e - \bar{\mathbf{B}}_e \bar{\mathbf{K}}_e & \bar{\mathbf{A}}_{12} - \bar{\mathbf{B}}_e \bar{\mathbf{K}}_{\tilde{e}} \\ 0 & \bar{\mathbf{A}}_{\tilde{e}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_e \\ \bar{\mathbf{x}}_{\tilde{e}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_e \\ 0 \end{bmatrix} \mathbf{r}.$$

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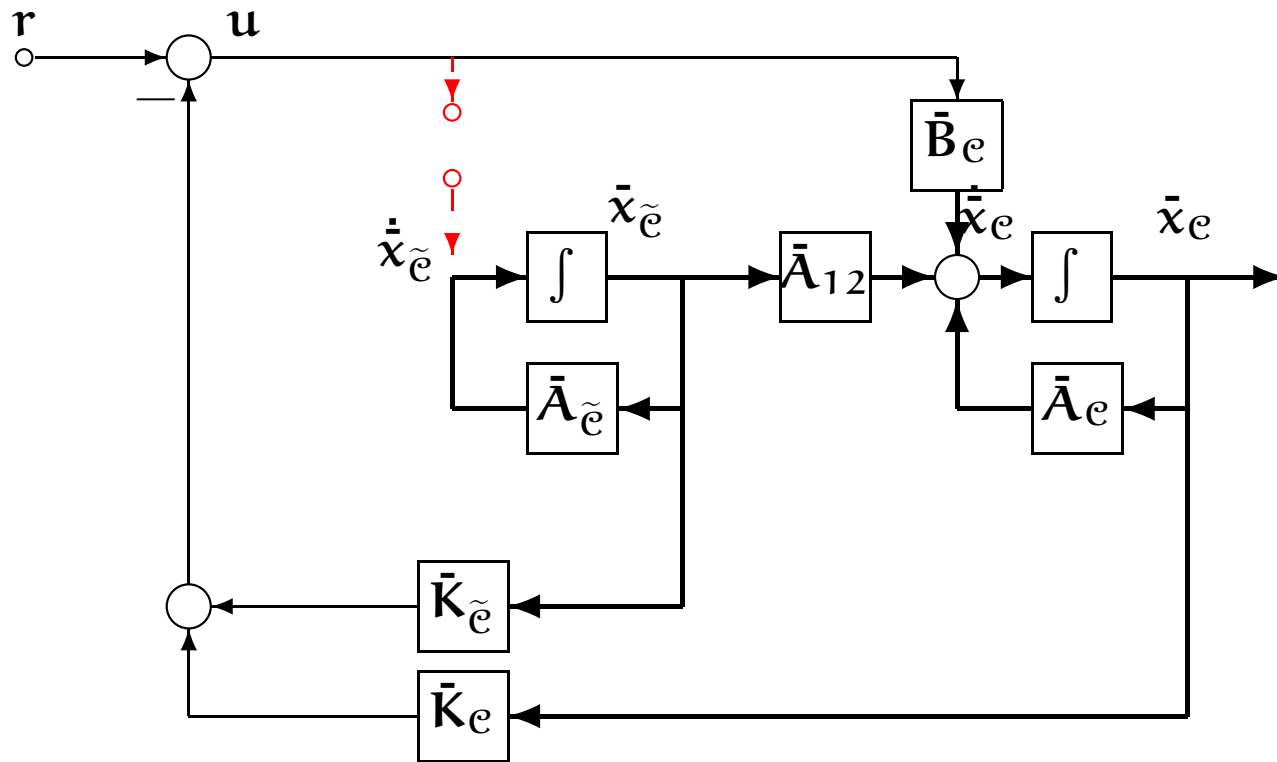
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We see that **the eigenvalues of $\bar{\mathbf{A}}_{\tilde{e}}$ are not affected by the state feedback**, so they remain **unchanged**.

The value of $\bar{\mathbf{K}}_{\tilde{e}}$ is **irrelevant** — the uncontrollable states cannot be affected.

State Feedback Stabilisation



State feedback in Controllable/Uncontrollable coordinates

- ▶ We conclude that the condition of **Controllability** is not only sufficient, but also necessary to place **all** eigenvalues of $\mathbf{A} - \mathbf{BK}$ in desired locations.

State Feedback Stabilisation

A notion of interest in control that is weaker than that of **Controllability** is that of **Stabilisability**.

Stabilisability. The system

$$\begin{aligned}\dot{\mathbf{x}}(\mathbf{t}) &= \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}\mathbf{u}(\mathbf{t}) \\ \mathbf{y}(\mathbf{t}) &= \mathbf{C}\mathbf{x}(\mathbf{t}),\end{aligned}$$

is said to be **stabilisable** if all its **uncontrollable states are asymptotically stable**.

This condition is equivalent to asking that the matrix $\bar{\mathbf{A}}_{\tilde{\mathbf{e}}}$ be **Hurwitz**.

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- ▶ If the system is **not controllable**, then we can apply **state feedback of the controllable states only** — uncontrollable states cannot be affected.
- ▶ A system is thus **stabilisable** if those states that are not controllable are already stable.