ELEC4410

# Control Systems Design

## *Lecture 23: Optimal LQG Control*

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# Outline

- LQG Control
- LQG Control for Disturbance Rejection

# Optimal LQ State Feedback (LQR)

Recall that the LQR problem considers the state space system

$$\begin{aligned} \dot{x} &= Ax + Bu, & x &\in \mathbb{R}^n, u \in \mathbb{R}^p \\ y &= Cx, & y &\in \mathbb{R}^q \end{aligned}$$

and the performance criterion

$$J = \int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt, \qquad \text{(J)}$$

where $Q$ is non negative definite and $R$ is positive definite. Then the optimal control minimising (J) is given by the **linear** state feedback law

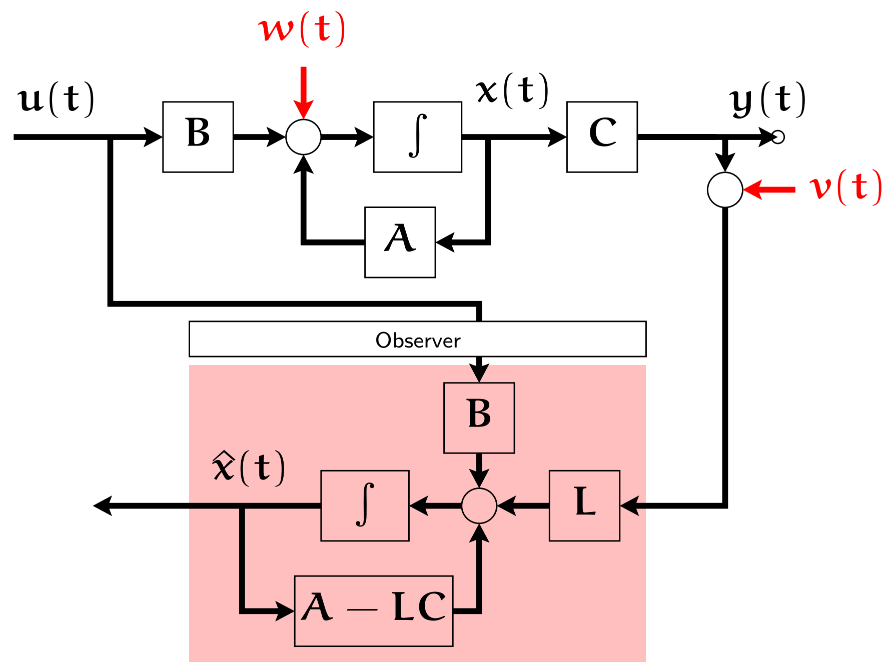
$$\boxed{u(t) = -Kx(t)} \quad \text{with} \quad \boxed{K = R^{-1}B^TP}$$

and where $P$ is the unique positive definite solution to the matrix **Algebraic Riccati Equation** (ARE)

$$\boxed{A^TP + PA - PBR^{-1}B^TP + Q = 0}$$

# Optimal State Estimation (LQE)

Recall that the dual optimal LQ estimator problem yields the **Kalman Filter**, which is the best possible estimator for the plant corrupted by noises $w$ and $v$




$$\begin{aligned} \dot{x} &= Ax + Bu + w \\ y &= Cx + v \end{aligned}$$

where $w$ and $v$ are **zero-mean stochastic Gaussian processes** uncorrelated in time and with each other, with covariances $E\left(ww^T\right) = W$ and $E\left(vv^T\right) = V$. The optimal estimator gain $L$ is $\boxed{L = PC^TV^{-1}}$ where $P$ is the solution to the algebraic Riccati equation

$$\boxed{AP + PA^T - PC^TV^{-1}CP + W = 0}$$

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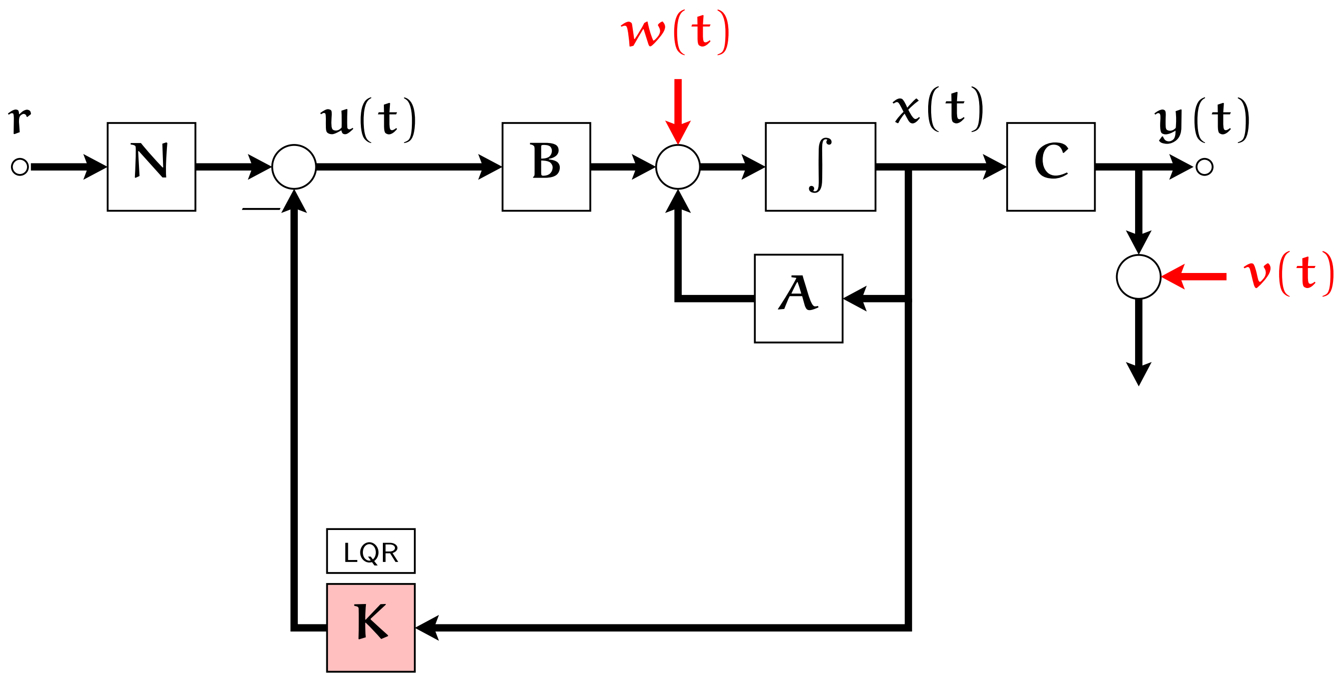
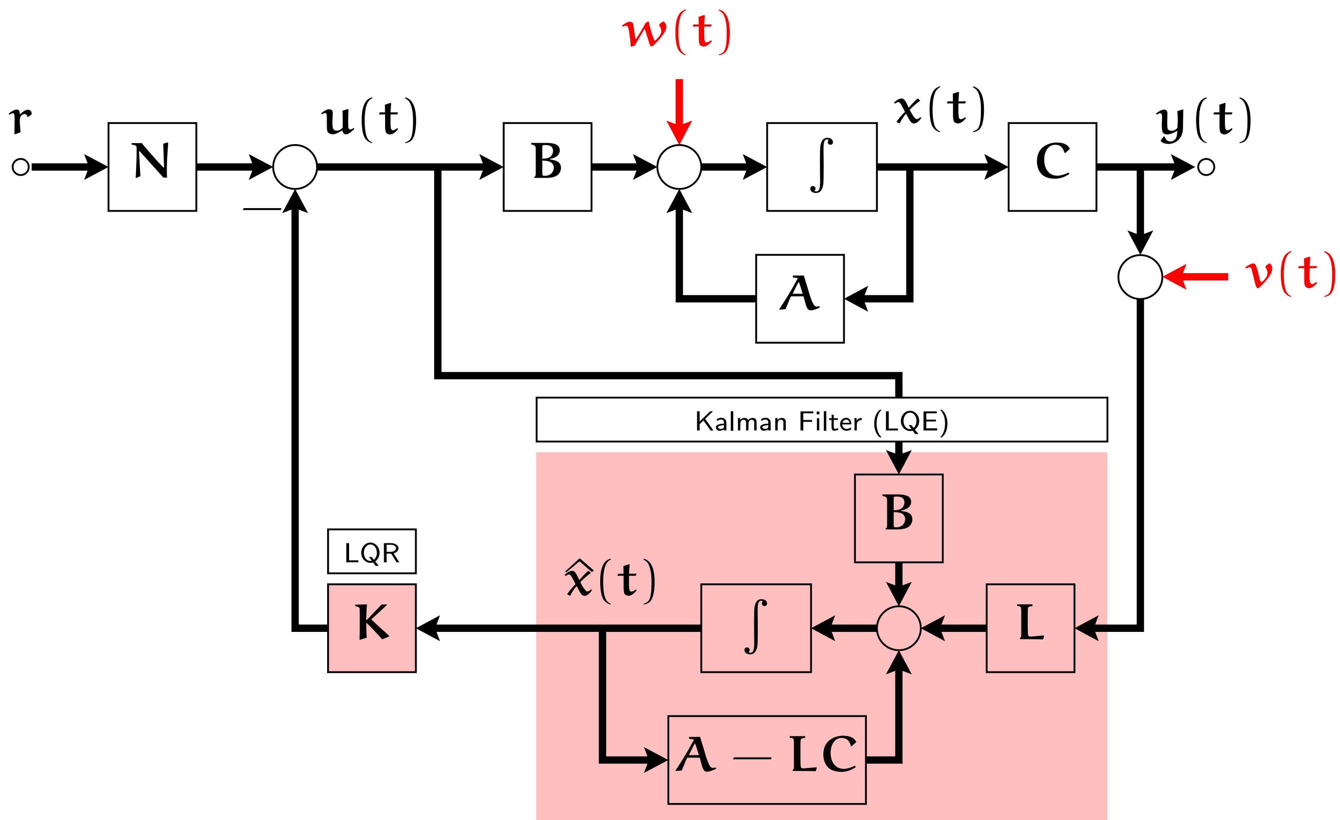


# *Linear Quadratic Gaussian (LQG) Control*

LQG is the optimal controller obtained as the combination of an optimal LQR state feedback gain with feedback from estimates from an optimal LQE state estimator.



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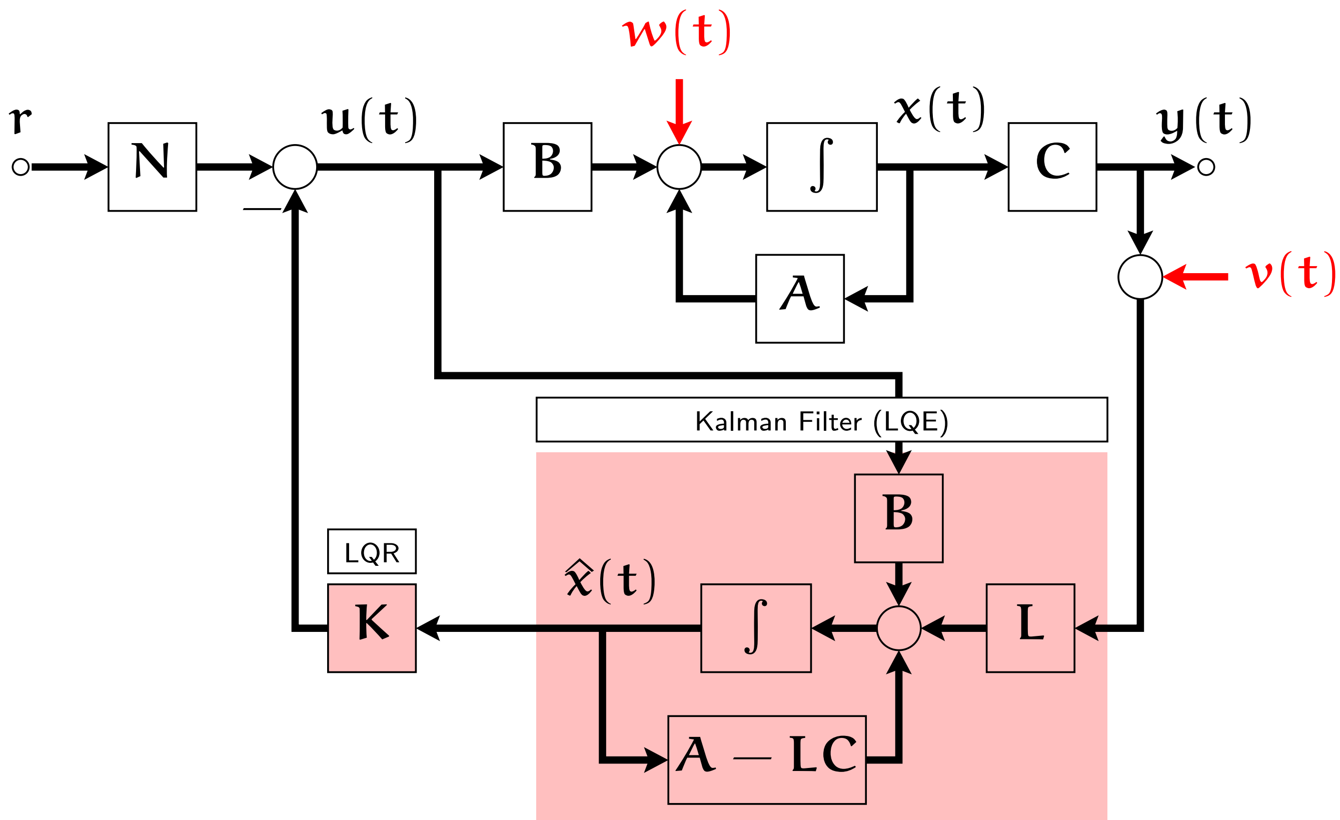


The **Separation Principle** allows us to design the LQR state feedback gain and the LQE independently.

# LQR, LQE and Separation Principle

The separation principle (or **certainty equivalence principle**) states that if we have a plant given by

$$\dot{x} = Ax + Bu + w$$
$$y = Cx + v$$

and we wish to design a controller to minimise

$$J = \lim_{T \to \infty} E\left\{ \frac{1}{T} \int_0^T \left[ x^T(t) Q x(t) + u^T(t) R u(t) \right] dt \right\}$$

then the optimal solution is given by combining the optimal LQ state feedback and optimal LQ observer given above.

Recall also from our treatment of pole assignment that the closed-loop poles are located at the eigenvalues of $A - BK$ and $A - LC$.

# *LQG Design Remarks*

- The combined controller including an LQR (optimal linear quadratic regulator) and LQE (optimal linear quadratic estimator) is usually called the **Linear Quadratic Gaussian (LQG)** controller.

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# LQG Design Remarks

- The combined controller including an LQR (optimal linear quadratic regulator) and LQE (optimal linear quadratic estimator) is usually called the **Linear Quadratic Gaussian (LQG)** controller.
- LQG can be used as a simple tool to get a ball-park controller with reasonable performance. Just as with pole assignment the plant must be augmented if features such as integral action are desired.
- There are also sophisticated design strategies based on LQG. These address not only intricate dynamics (e.g. resonant systems or interactions in multivariable systems) but also ensure the resultant design has suitable **robustness properties**. Such strategies are beyond the scope of this course, but the book "Multivariable Feedback Design" by J.M. Maciejowski (Addison Wesley, 1989) is recommended.

# *LQG Control for Disturbance Rejection*

As an LQG control example we consider an application of **disturbance rejection by the Internal Model Principle** (from Bay, *Linear State Space Systems*, McGraw-Hill, 1999).

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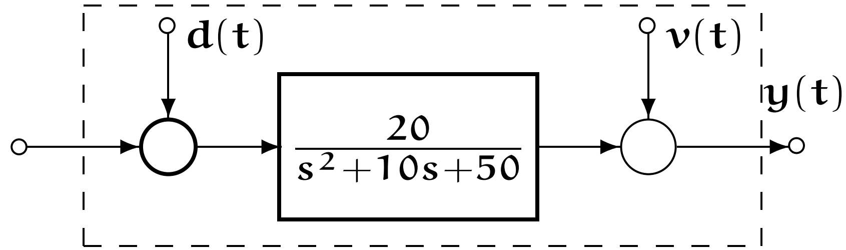
Consider the plant




$$\dot{x} = \begin{bmatrix} -10 & -50 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 20 & 0 \end{bmatrix} x.$$

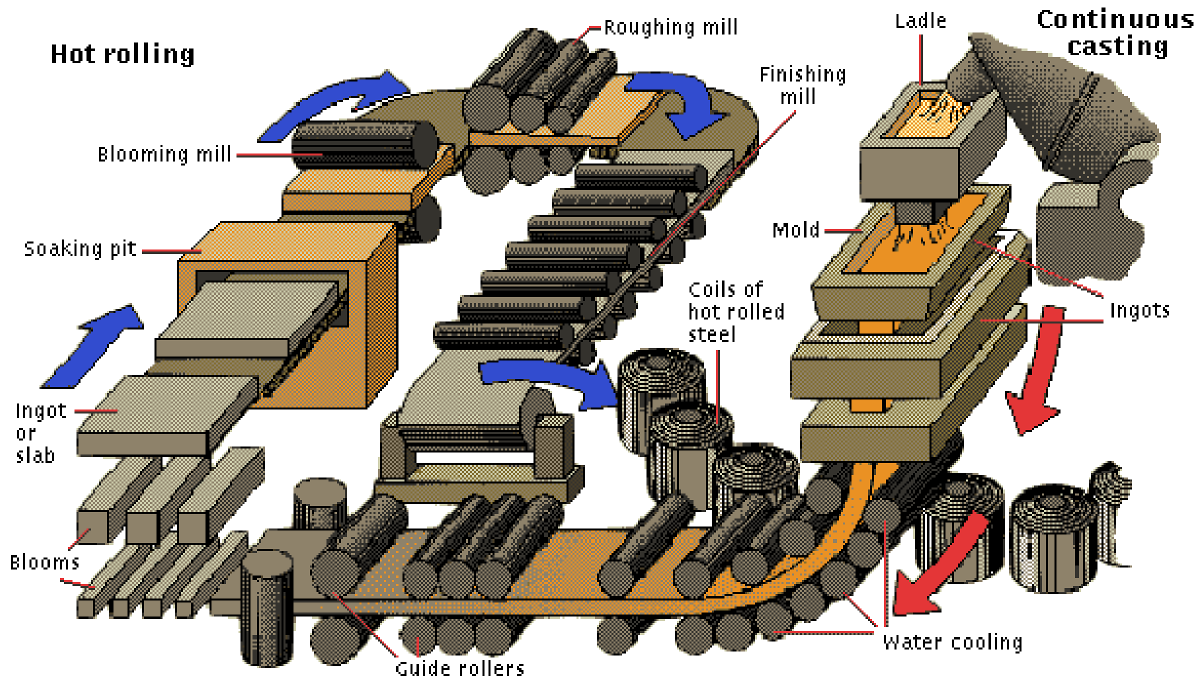
It is known that the output of this plant is corrupted by a zero mean additive white noise with covariance $0.005$.

It is also known that, in operation, the state vector of this plant is corrupted by a narrow-band noisy signal of approximately 1Hz.

The goal of the control problem is to reject this 1-Hz disturbance.

# LQG Control for Disturbance Rejection

Such a disturbance rejection problem is found for example in the mould level control problem in a continuous-casting machine.




The regulation of the mould level is important, since it affects the quality of the casted blooms. However, the mould must be affected of a periodic movement to prevent the metal sticking to its walls. Such movement induces a slow periodic disturbance.

# *LQG Control for Disturbance Rejection*

A basic principle we will follow is that in order to be able to reject a disturbance, or track a reference, we need to incorporate **a model** of the disturbance in the controller. This is known as the **Internal Model Principle**.

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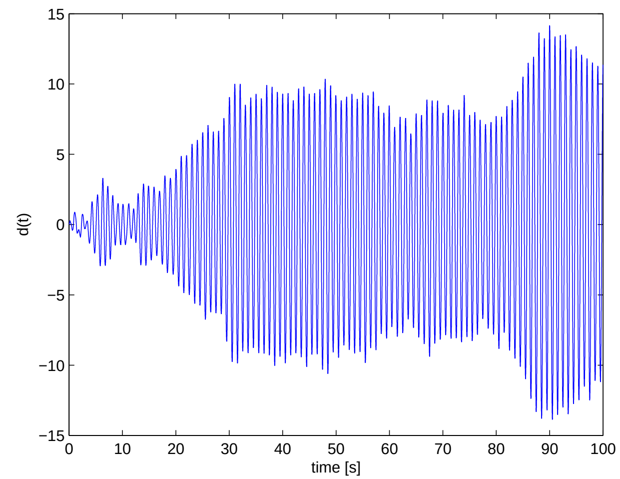
One familiar example of disturbance rejection (and reference tracking) based on the Internal Model Principle is the **integral action** in state feedback: In order to reject constant disturbances we **augment** their model (an integrator) into the plant.

# LQG Control for Disturbance Rejection

For the additive noise at the output, we model it as output disturbance noise signal $v$ with variance $V = 0.005$.

In the case of the 1-Hz disturbance, we model it by an auxiliary state equation, where $w$ is a zero-mean white noise with covariance $W = 0.01$,

$$\dot{x}_d = \overbrace{\begin{bmatrix} 0 & 2\pi \\ -2\pi & 0 \end{bmatrix}}^{A_n} x_d + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B_n} w$$
$$y_d = \underbrace{\begin{bmatrix} 100 & 0 \end{bmatrix}}_{C_n} x_d$$




This (coloured) noise excites the system to produce a signal of 1Hz, scaled by a factor of 100.

# LQG Control for Disturbance Rejection

We will treat the noisy output as a coloured input disturbance noise entering at the iput of the original system,

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -10 & -50 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (u + d) &&= Ax + B(u + d) \\
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We will design a Kalman filter to reject this disturbance. For the design we combine plant and disturbance in the **augmented plant** $(A_a, B_a, C_a, D_a)$:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_d \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & A_n \end{bmatrix} \begin{bmatrix} x \\ x_d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ B_n \end{bmatrix} w$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_d \end{bmatrix} + v$$

# LQG Control for Disturbance Rejection

Now we design the gain $\mathbf{L}$ of our Kalman filter

$$\dot{\hat{x}} = (\mathbf{A}_a - \mathbf{L}\mathbf{C}_a)\hat{x} + \mathbf{B}_a u + \mathbf{L}y$$

by using our hypothetical plant model (the augmented plant) and the statistical properties of the noises $v$ and $w$,

$$V = \mathsf{E}\{v^2(t)\} = 0.005, \quad W = \mathsf{E}\{w^2(t)\} = 0.01.$$

In MATLAB we can use the function `lqe`

```
L = lqe(Aa,[0*B;Bn],Ca,W,V);
```

We obtain

$$\mathbf{L} = \begin{bmatrix} 0.9037 \\ 8.1671 \\ 1.4021 \\ 0.1848 \end{bmatrix}.$$

# LQG Control for Disturbance Rejection

In other words, we have modelled the disturbance as the output of an hypothetical plant, and incorporated it to the model of the real plant to obtain the augmented model

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\dot{x}_a &= A_a x_a + B_a u + E w \\
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Now, to control this system, we will compute a state feedback gain $u = -Kx_a$ minimising the performance criterion

$$J = \int_0^\infty \left[ x_a^T(t) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x_a(t) + u^T(t) R u(t) \right] dt$$

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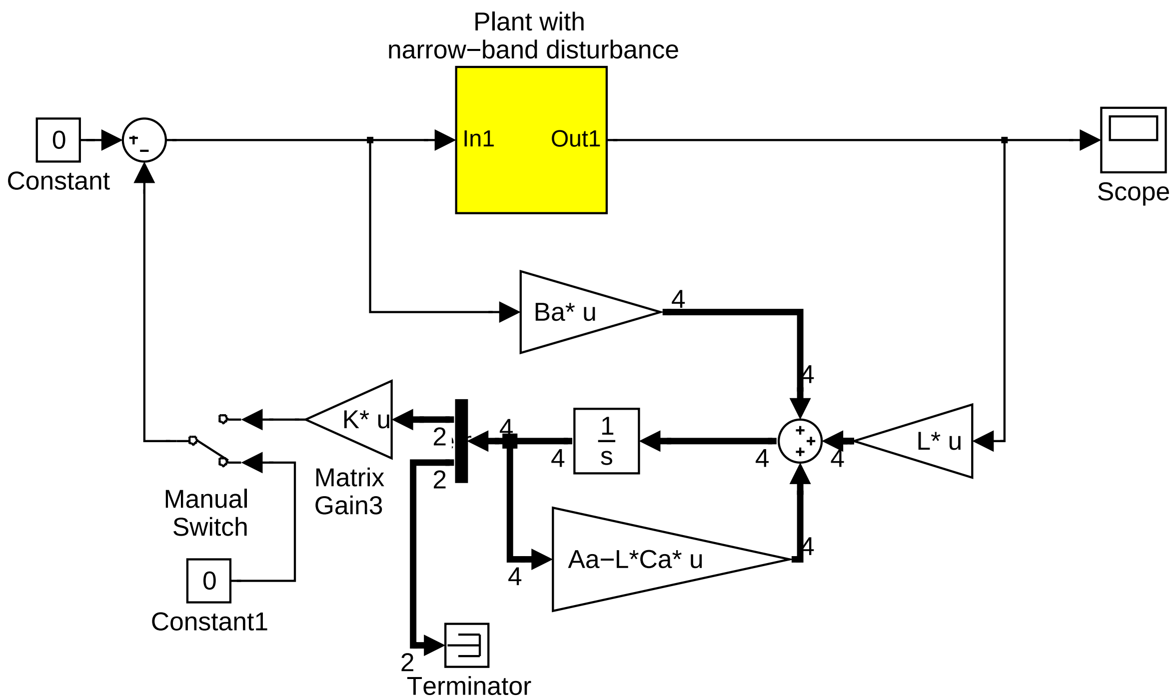
Note that this form of $Q$ penalises only the real system output — we don't penalise the disturbance "states", which are hypothetical and uncontrollable anyway. We will use two different values of $R > 0$ to see the effect of each.

# *LQG Control for Disturbance Rejection*

The trick in rejecting the narrow-band disturbance is not in trying to **control** it, since it is uncontrollable by definition, but in **filtering** it by using its model in the Kalman filter

$$\dot{x}_a = (A_a - LC_a)x_a + B_a u + Ly$$



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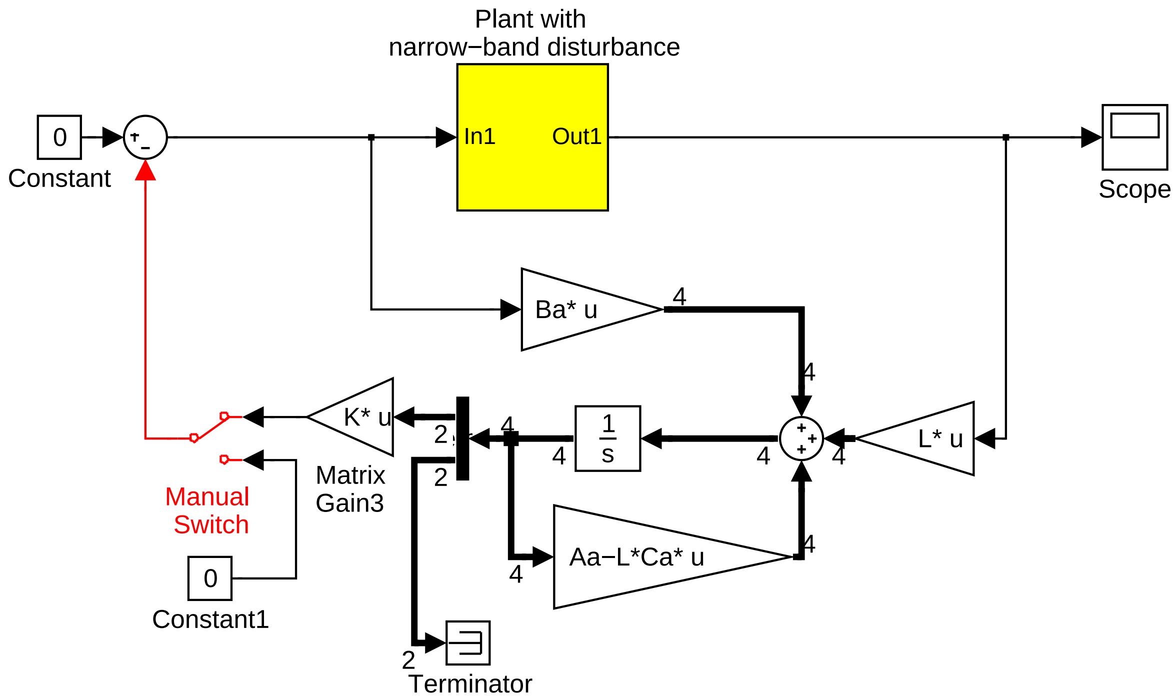
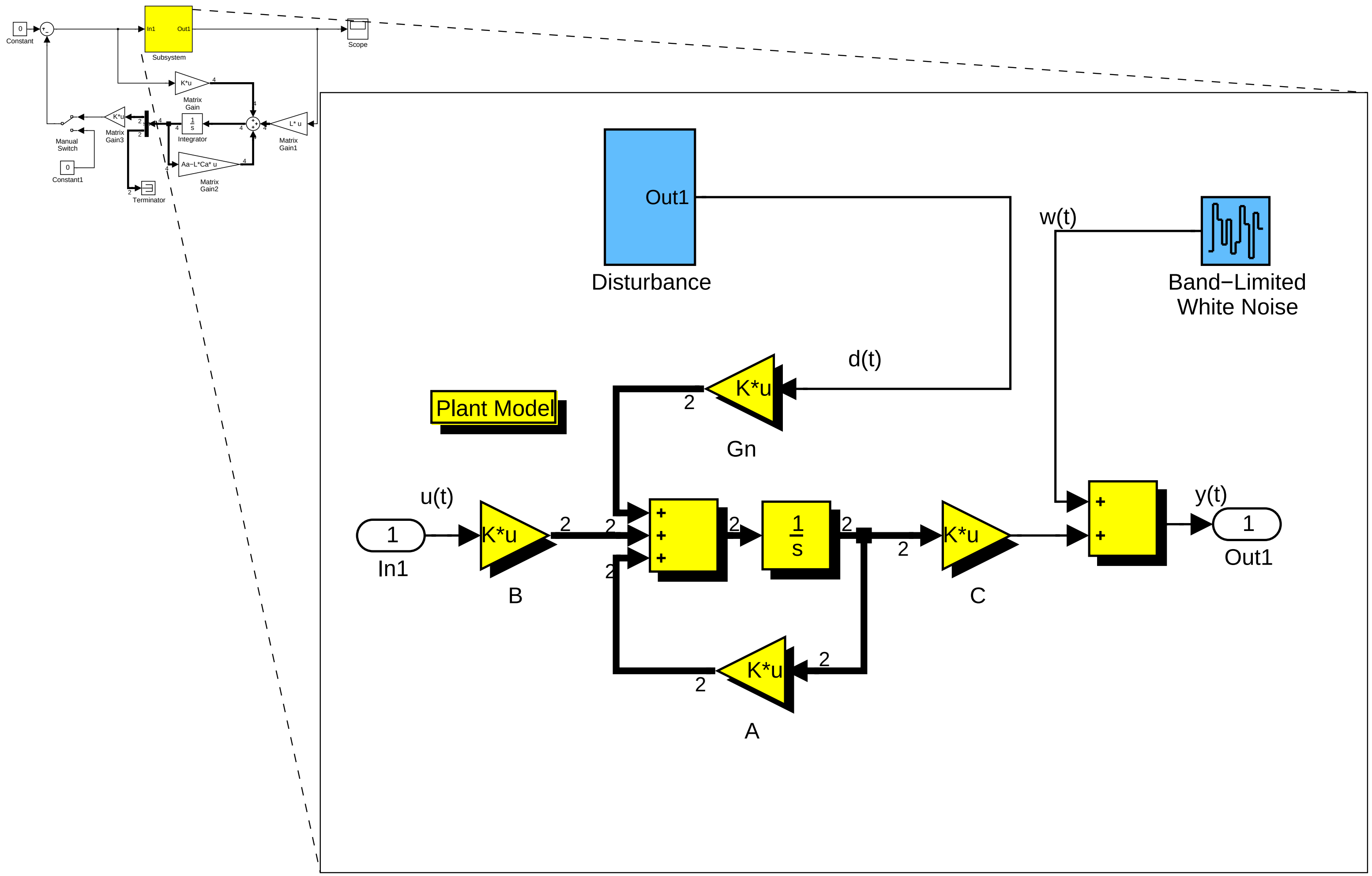



The estimator gain $L$ is computed as to optimally minimise the effect of the white noises $w$ and $v$ using the noise covariances $W$ and $V$.

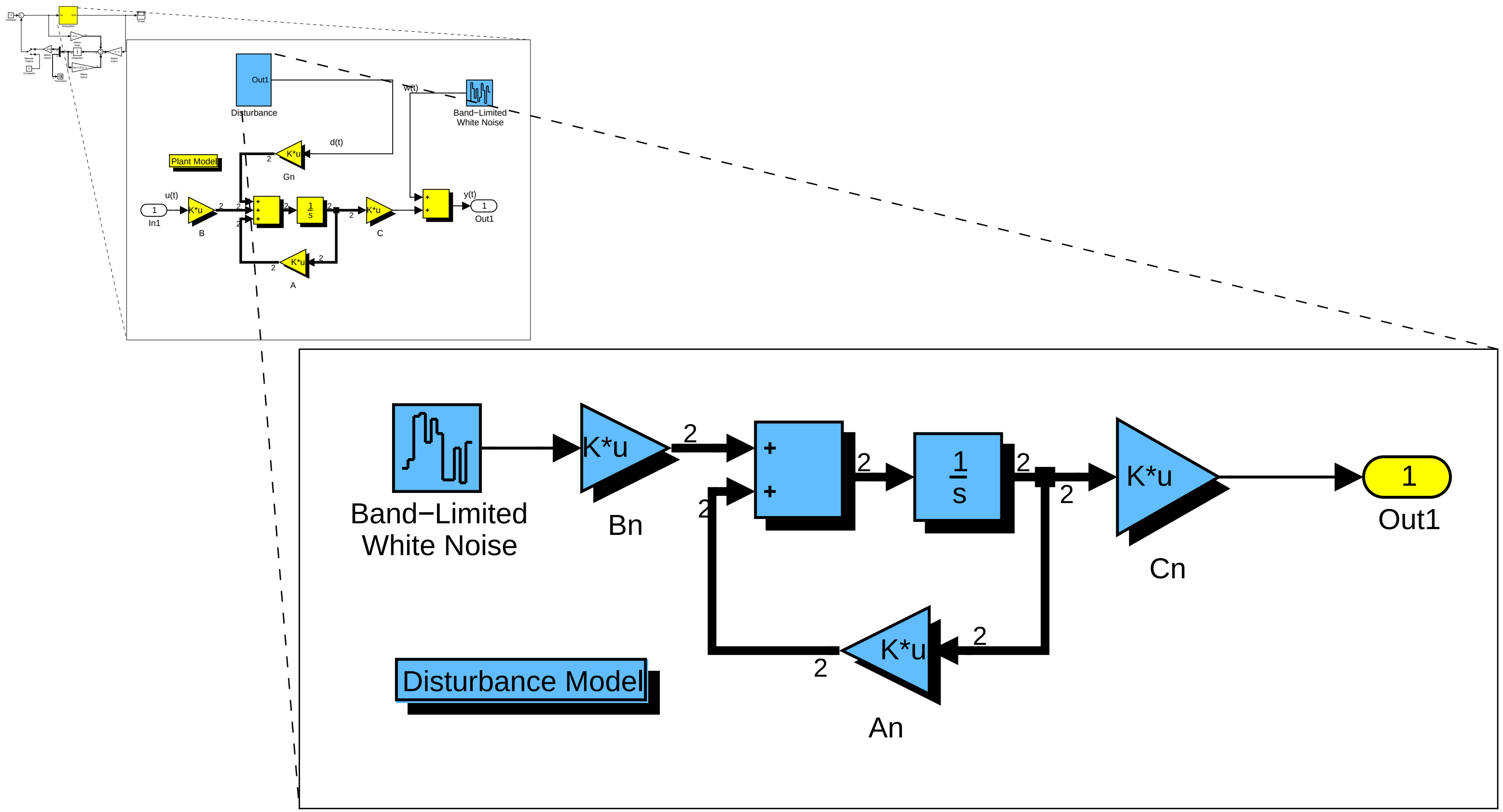
# LQG Control for Disturbance Rejection

The SIMULINK diagram below shows the detail of the augmented plant implementation.



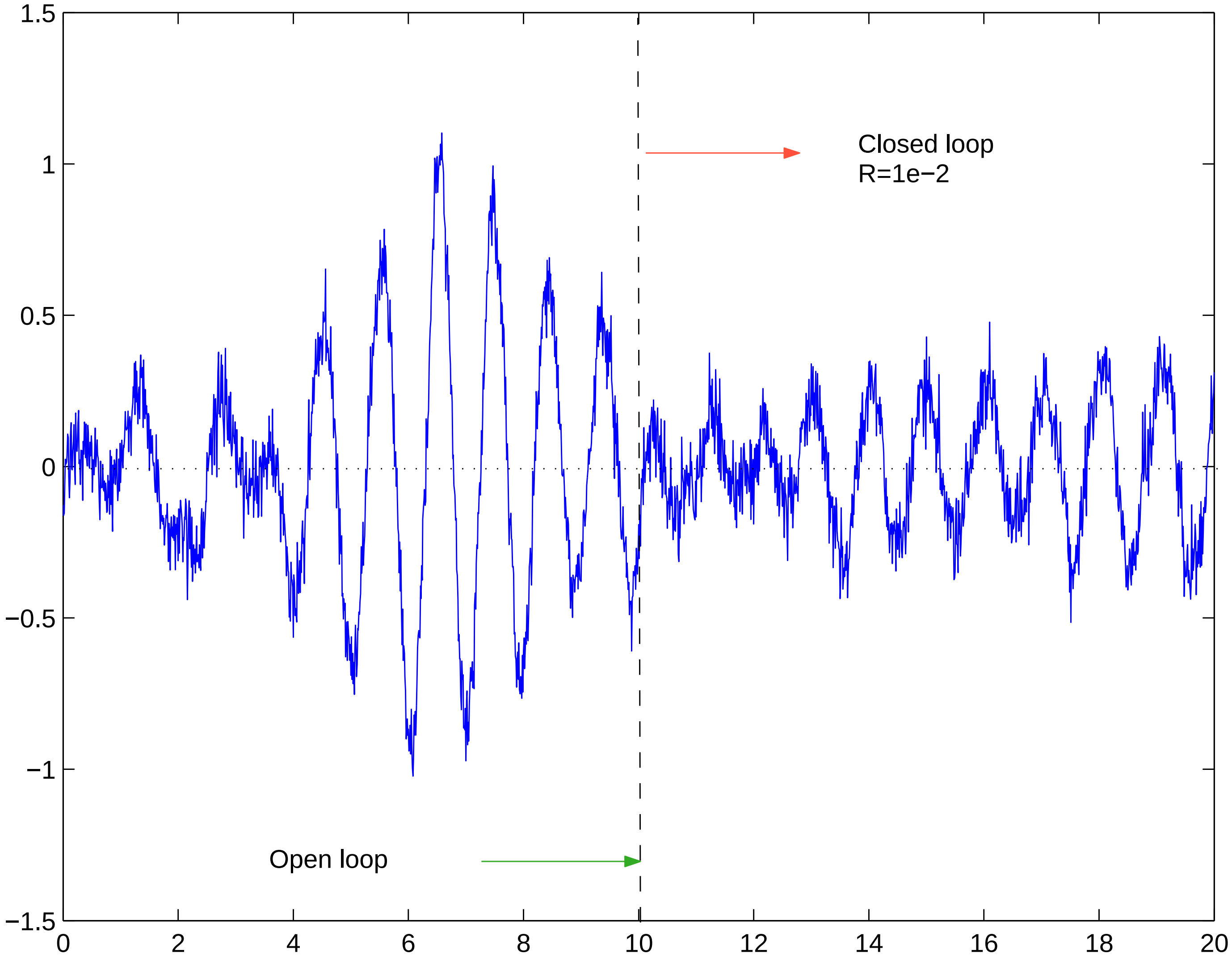
# LQG Control for Disturbance Rejection

This SIMULINK diagram shows how we implement the narrow-band input disturbance model.



# LQG Control for Disturbance Rejection

We show the output of the system in **open-loop** ($\mathbf{K} = \mathbf{0}$) and in **closed-loop** for different values of $\mathbf{R}$.



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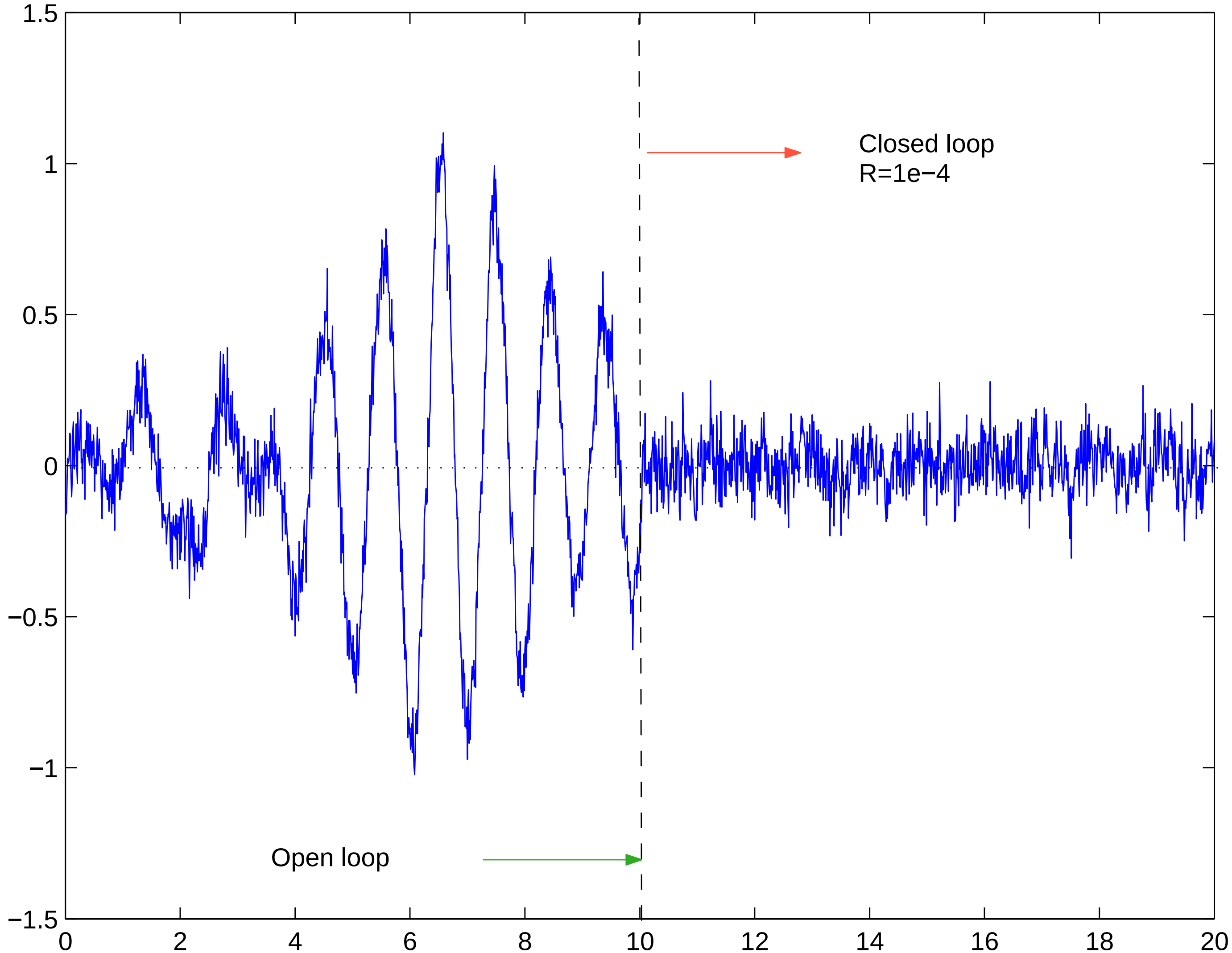

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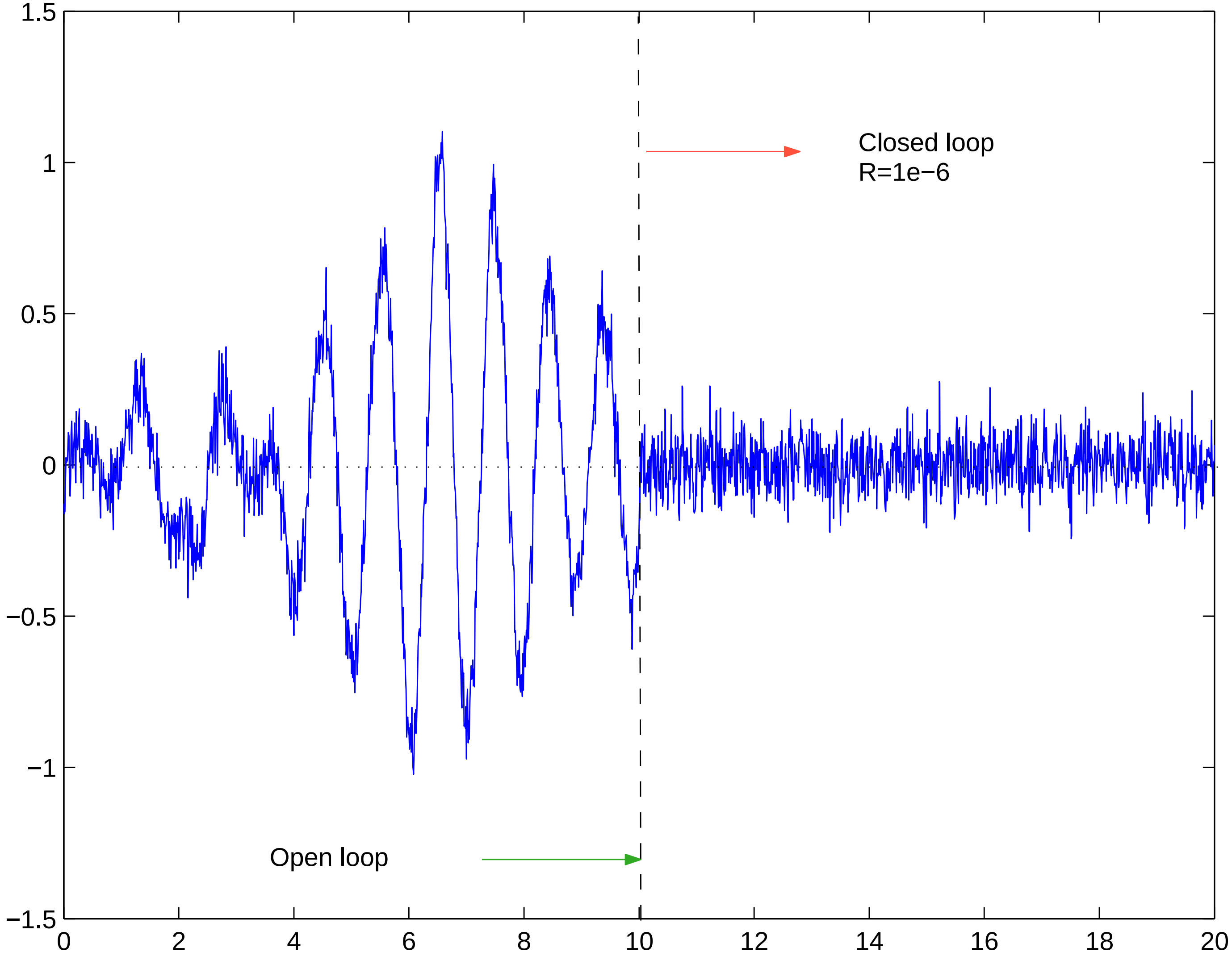
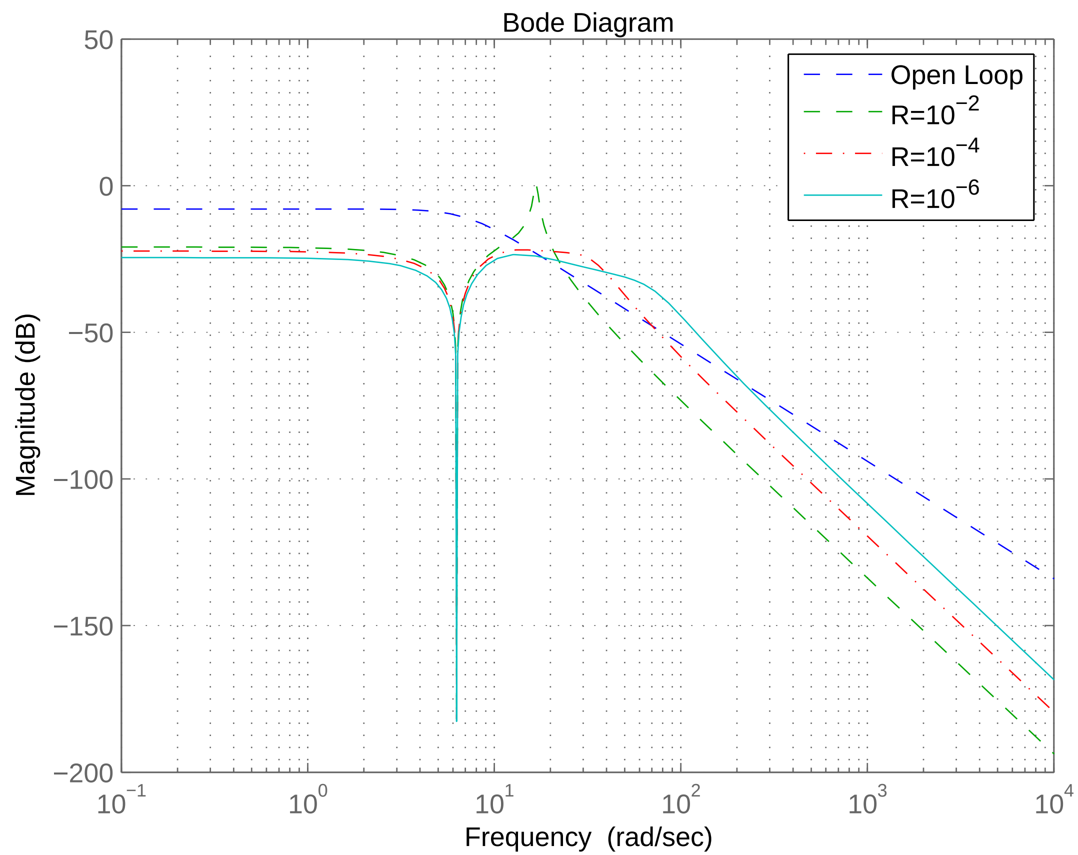


# LQG Control for Disturbance Rejection

- For values of the control weight $\mathbf{R} < 10^{-4}$ there is no significant improvement in the response, because the system has a "floor" noise level due to the measurement noise $\boldsymbol{v}$ (which we will not be able to remove from the output).
- However, we could effectively improve the response in removing the narrow band disturbance.
- This is a way of dealing with disturbances **in the observer** rather than in the feedback control, as we did in the integral action scheme we learned previously.

# LQG Control for Disturbance Rejection

The plot shows the magnitude Bode plot of the closed loop system transfer function from the input disturbance $\mathbf{d}$ to the output for the different values of $\mathbf{R}$, and in open loop.




We can see how the LQG control produces a closed loop system with a notch at the frequency of the disturbance (1Hz).

# Final Remarks on LQG

- Of course, we can incorporate in our LQG design integral action, by augmenting the plant with the integral of the tracking error. We should then be careful in selecting $\mathbf{Q}$ so that we penalise the tracking error and not the state that is suppose to track a reference.

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- LQG is one of the most basic and important tools in the control engineer's toolbox. For a state space models, it constitutes in most cases the first choice for control design.
- LQG is generally not robust, but it will almost always give good results when the model of the system is reasonably accurate.