FEL3210 Multivariable Feedback Control

Lecture 5: Uncertainty and Robustness in SISO Systems
[Ch.7-(8)]

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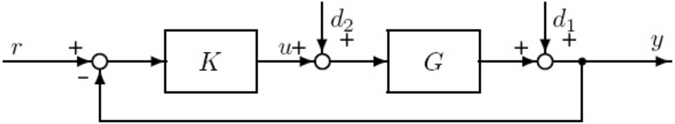
Outline

- Defining robust stability and robust performance
- Uncertainty descriptions
- Robust stability from Nyquist
- Robust stability from Small Gain Theorem
- Robust performance from Nyquist

Nominal stability and performance - NS / NP



$G(s)$ is a model!

- the closed-loop system satisfies **nominal stability (NS)** iff

$$S = (I + GK)^{-1} \; ; \quad KS \; ; \quad S_I = (I + KG)^{-1} \; ; \quad GS_I$$

all have all poles in the complex LHP

- the closed-loop system satisfies **nominal performance (NP)** e.g., if

$$\left\| \begin{matrix} W_P S \\ W_T T \\ W_u KS \end{matrix} \right\|_\infty < 1$$

Robustness

A control system is robust if it is **insensitive** *to differences between the true system and the model of the system that was used to design the controller. These differences are called* **model/plant mismatch** *or* **model uncertainty**

Model uncertainty

- sources of model uncertainty:
  - parametric uncertainty
  - neglected dynamics
  - unmodelled dynamics
  - (nonlinearities)
- represent system not by a single model $G(s)$, but by a **model set** Π that covers all possible models within the uncertainty description

$$G(s) \in \Pi \quad \wedge \quad G_{true}(s) \in \Pi$$

$G(s)$ - nominal model, $G_{true}(s)$ - true system

Robust stability and performance - RS / RP

let $G_p(s)$ denote any model in the model set Π

- the closed-loop system satisfies **robust stability (RS)** iff

$$S_p = (I + G_p K)^{-1} \ ; \ \ KS_p \ ; \ \ S_{Ip} = (I + KG_p)^{-1} \ ; \ \ G_p S_I$$

all have all poles in the complex LHP for all $G_p \in \Pi$

- the closed-loop system satisfies **robust performance (RP)** e.g., if

$$\left\| \begin{matrix} W_P S_p \\ W_T T_p \\ W_u S_p \end{matrix} \right\|_\infty < 1$$

for all $G_p \in \Pi$

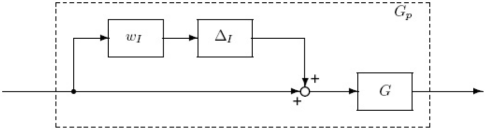
This lecture

1. determining the model set Π
2. analysing **RS** and **RP**, given Π

focus on SISO systems (MIMO next time)

Classes of uncertainty

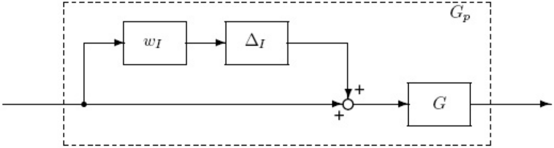
- **Parametric uncertainty:** model structure and order is known, but some parameters are uncertain
- **Unmodelled and neglected dynamics:** model does not describe complete dynamics of system, and order of system is unknown. In particular, dynamics at high frequencies is usually not described completely due to lack of knowledge of system behavior at these frequencies.
- **Lumped uncertainty:** combine several sources of uncertainty into a perturbation of a chosen model structure



Lumped uncertainty descriptions

- **Multiplicative uncertainty:**

$$\Pi_I : \quad G_p(s) = G(s)(1 + w_I(s)\Delta_I(s)) \; ; \quad \|\Delta_I\|_\infty \leq 1$$



  - the uncertainty weight w_I describes frequency dependence of uncertainty
  - the perturbation $\Delta_I(s)$ is any <u>stable</u> transfer-function with magnitude less than one for all frequencies.

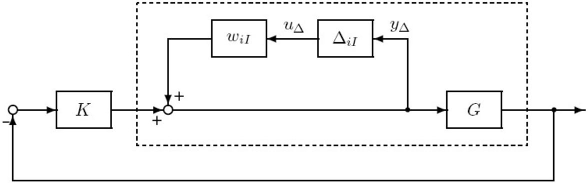
Examples of allowable Δ_I's

$$\frac{s-z}{s+z} \; ; \quad e^{-\theta s} \; ; \quad \frac{1}{(\tau s + 1)^n} \; ; \quad \frac{0.1}{s^2 + 0.1s + 1}$$

Lumped uncertainty descriptions

- **Inverse multiplicative uncertainty:**

$$\Pi_{iI}: \quad G_p(s) = G(s)(1 + w_{iI}(s)\Delta_{iI}(s))^{-1} \; ; \quad \|\Delta_{iI}\|_\infty \le 1$$

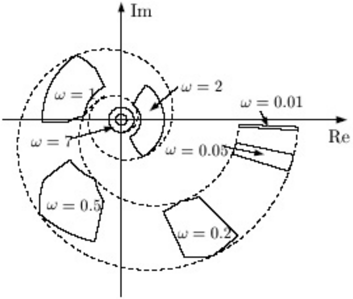


  – allows for uncertain number of RHP poles even if $\Delta_{iI}(s)$ is required to be stable

Uncertainty in the frequency domain

Example: parametric uncertainty

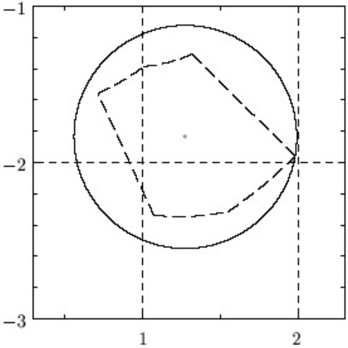
$$G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \; 2 \leq k, \tau, \theta \leq 3$$



At each frequency, a region of complex numbers $G_p(j\omega)$ is generated when the model parameters are varied within their uncertainty region

Disc approximation

Approximate the uncertainty region by a circular disc at each frequency ω, with center at nominal model $G(j\omega)$

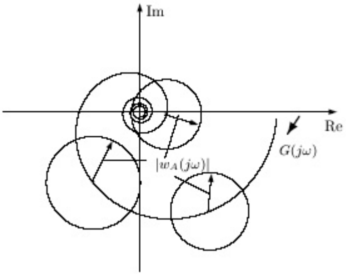


– introduces some conservatism, i.e., include models not in the original set.

Disc approximation from complex perturbation

Discs with radius $|w_A(j\omega)|$ are generated from

$$\Pi_A : G_p(s) = G(s) + w_A(s)\Delta_A(s) \; ; \quad |\Delta_A(j\omega)| < 1 \; \forall\omega$$

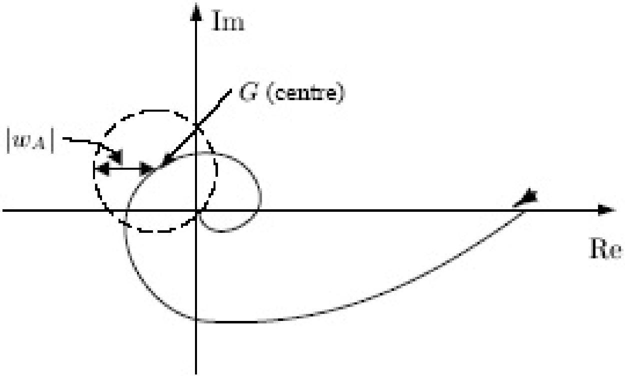


- Note: Π_I can be obtained from Π_A through

$$|w_I(j\omega)| = \frac{|w_A(j\omega)|}{|G(j\omega)|}$$

100 % uncertainty

At frequencies where $|w_A(j\omega)| > |G(j\omega)|$, or equivalently, $|w_I(j\omega)| > 1$, we have no knowledge about phase of system



– require bandwidth to be less than frequency where $|w_I(j\omega)| = 1$

Obtaining the uncertainty weight

1. Decide on a nominal model $G(s)$
2. *Additive uncertainty:* at each frequency determine the smallest radius $l_A(\omega)$ which includes all possible plants in Π

$$|w_A(j\omega)| \geq l_A(\omega) = \max_{G_p \in \Pi} |G_p(j\omega) - G(j\omega)|$$

3. *Multiplicative uncertainty:* at each frequency determine the largest relative distance $l_I(\omega)$

$$|w_I(j\omega)| \geq l_I(\omega) = \max_{G_p \in \Pi} \left| \frac{G_p(j\omega) - G(j\omega)}{G(j\omega)} \right|$$

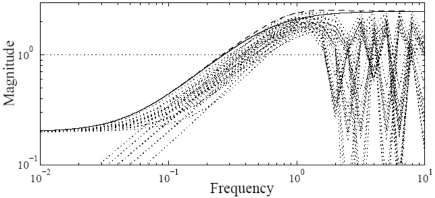
Example: multiplicative weight

$$\Pi : \quad G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \; 2 \le k, \tau, \theta \le 3$$

1. choose delay-free nominal model

$$G(s) = \frac{\bar{k}}{\bar{\tau} s + 1} = \frac{2.5}{2.5s + 1}$$

2. generate frequency response $|G_p - G|/|G|$ for all allowed parameters



3. fit upper bound

Choice of nominal model

Three options for choice of nominal model $G(s)$

1. **simple model:** low-order and delay-free
   - $(+)$ simplifies controller design, $(\div)$ potentially large uncertainty
2. **mean parameter model:** use average parameter values
   - $(+)$ simple choice, smaller uncertainty region than with 1., $(\div)$ not optimal
3. **central frequency response:** use model that yields the smallest uncertainty disc at each frequency
   - $(+)$ smallest uncertainty, $(\div)$ complex procedure and high order model

Neglected dynamics as uncertainty

Assume full model

$$G(s) = G_1(s)G_2(s)$$

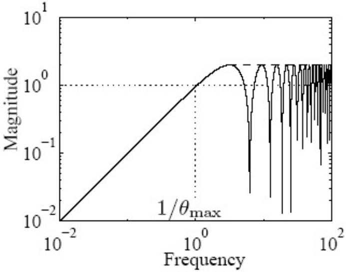
We want to neglect $G_2(s)$ in our nominal model. Then

$$l_I(\omega) = \max_{G_p} \left| \frac{G_p - G_1}{G_1} \right| = \max_{G_2(s) \in \Pi_2} |G_2(j\omega) - 1|$$

where Π_2 denotes that the neglected dynamics may be uncertain

Example:

neglected delay $G_2(s) = e^{-\theta s}, \ \theta \in [0, \theta_{max}]$



$$l_I(\omega) = \begin{cases} \left|1 - e^{-j\omega\theta_{max}}\right| & \omega \leq \pi/\theta_{max} \\ 2 & \omega \geq \pi/\theta_{max} \end{cases}$$

$$\Rightarrow \quad w_I(s) = \frac{\theta_{max} s}{\frac{\theta_{max}}{2} s + 1}$$

Unmodelled dynamics as uncertainty

Unmodelled dynamics are dynamics we have neglected simply because we have no knowledge about it, e.g., true system order.

Usually, represent unmodelled dynamics by some simple multiplicative weight

$$w_I(s) = \frac{\tau s + r_0}{(\tau / r_\infty)s + 1}$$

Three parameters

- r_0 is relative uncertainty at low frequencies
- at $\omega = 1/\tau$, relative uncertainty is $\sim 100\%$
- r_∞ is relative uncertainty at high frequencies

Next...

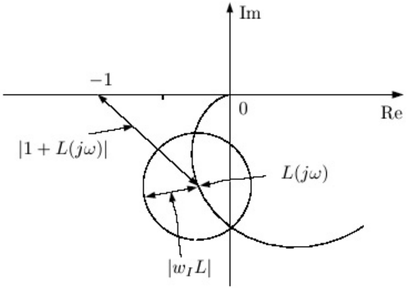
- Robust stability from Nyquist
- Robust stability from *Small Gain Theorem*
- Robust performance from Nyquist

Robust stability (RS) from Nyquist plot

Consider SISO system with multiplicative uncertainty

$$\Pi_I : \quad L_p = G_p K = GK(1 + w_I \Delta_I) = L + w_I L \Delta_I, \quad \|\Delta_I\|_\infty \leq 1$$

Assume open loop $L_p(s)$ is stable, then for robust closed-loop stability the Nyquist plot of $L_p(j\omega)$ should not encircle the point -1 for any $G_p \in \Pi_I$



$$RS \quad \Leftrightarrow \quad |w_I L| < |1 + L|, \quad \forall \omega$$

$$\Updownarrow$$

$$\left| \frac{w_I L}{1 + L} \right| < 1, \ \forall \omega \quad \Leftrightarrow \quad \|w_I T\|_\infty < 1$$

– necessary and sufficient condition for RS

Small Gain Theorem - Linear systems

Theorem 4.12 *consider a feedback loop with a stable loop transfer-function* $L(s)$. *Then the closed-loop is stable if*

$$\|L(j\omega)\| < 1 \quad \forall\omega$$

where $\|\cdot\|$ *denotes any matrix norm*

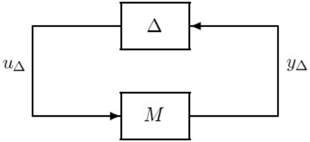
"Proof"

- generalized Nyquist criterion: for closed-loop instability some $\lambda_i(L(j\omega))$ should encircle -1, i.e., there must exist some i and ω such that $\lambda_i(L(j\omega)) = -A$ with $A > 1$.
- thus, $\rho(L(j\omega)) > |A| > 1$ for some ω. Hence, we can not have closed-loop instability if $\rho(L) < 1\ \forall\omega$.
- since $\rho(L) \leq \|L\|$ for any matrix norm, the result follows

- sufficient, but not necessary, condition

RS from small gain theorem

Write closed-loop on the form



- closed-loop stable if $M(s)$ and $\Delta(s)$ stable and $\|M\Delta\|_\infty < 1$
- with $G_p = G(I + w_I\Delta_I)$ we derive

$$M = w_I KG(I + KG)^{-1} = w_I T$$

where the last equality holds for SISO systems

- thus, with assumption that $\Delta(s)$ stable we get RS condition

$$RS \quad \Leftrightarrow \quad \|w_I T\|_\infty < 1$$

same result as with Nyquist, but no need to assume $L_p(s)$ stable

Some Technicalities

- If $\|\Delta\|_\infty \leq 1$ (as above) then the condition

$$\|w_I T\|_\infty < 1$$

  - is necessary and sufficient if $G(s)$ and $K(s)$ lack poles on the imaginary axis.
  - is only sufficient if $G(s)$ and/or $K(s)$ have poles on the imaginary axis.

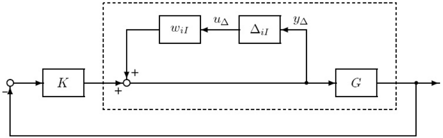
- If $\|\Delta\|_\infty < 1$ then the condition

$$\|w_I T\|_\infty \leq 1$$

  is necessary and sufficient for all $G(s)$ and $K(s)$

Proof: see e.g., Zhou, Doyle and Glover, p. 223

RS for inverse multiplicative uncertainty



$$\Pi_{iI}: \quad G_p = G(1 + w_{iI}\Delta_{iI})^{-1}\ ; \quad \|\Delta_{iI}\|_\infty \leq 1$$

- on $M - \Delta$-form we derive

$$M = (I + KG)^{-1} w_{iI} = w_{iI} S$$

  where the last equality holds for SISO systems
- thus, robust stability if Δ_{iI} stable and

$$RS \quad \Leftrightarrow \quad \|w_{iI} S\|_\infty < 1$$

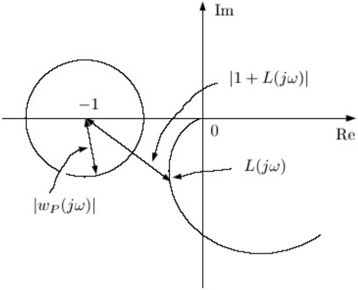
- condition corresponds to $|S| < \frac{1}{|w_{iI}|}\ \forall\omega$, thus need tight control, $|S|$ small, where uncertainty $|w_{iI}|$ large.

Robust performance (RP) in SISO systems

Consider first nominal performance requirement

$$NP \quad \Leftrightarrow \quad \|w_P S\|_\infty < 1 \quad \Leftrightarrow \quad |w_P| < |1 + L| \ \forall \omega$$

In Nyquist



- "avoid -1 with some margin $|w_P(j\omega)|$"

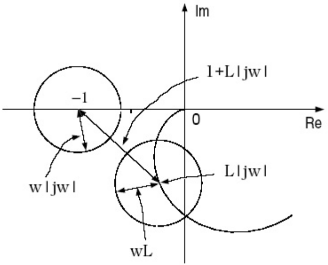
Robust performance

Robust performance requirement

$$RP \quad \Leftrightarrow \quad \|w_P S_p\|_\infty < 1 \;\; \forall S_p \quad \Leftrightarrow \quad |w_P| < |1 + L_p| \;\; \forall L_p, \forall \omega$$

With $L_p = L(1 + w_I \Delta_I) = L + w_I L \Delta_I$ we get

$$RP \quad \Leftrightarrow \quad |w_P| + |w_I L| < |1 + L|$$



$$|w_P(1+L)^{-1}| + |w_I L(1+L)^{-1}| < 1 \;\; \forall \omega \quad \Leftrightarrow \quad |w_P S| + |w_I T| < 1 \;\; \forall \omega$$

RP condition - SISO systems

$$RP \quad \Leftrightarrow \quad \max_{\omega} \left(|w_P S| + |w_I T| \right) < 1 \;\; \forall \omega$$

- NP ($|w_P S| < 1$) and RS ($|w_I T| < 1$) prerequisites for RP
- if NP and RS satisfied then

  $$\max_{\omega} \left(|w_P S| + |w_I T| \right) \leq 2 \max\{|w_P S|, |w_I T|\} < 2$$

  thus, with a factor of at most 2 we get "RP for for free" when NP and RS are satisfied
- the H_∞-norm bound

  $$\left\| \begin{pmatrix} w_P S \\ w_I T \end{pmatrix} \right\|_\infty = \max_{\omega} \sqrt{|w_P S|^2 + |w_I T|^2} < 1$$

  deviates from RP condition by a factor of at most $\sqrt{2}$.
  Thus, for SISO systems the RP condition can essentially be formulated as an H_∞-problem

Next time

MIMO systems: uncertainty Δ is a matrix

- NP and RS conditions similar to SISO case for full block uncertainty (Δ full matrix)
- need special tool for structured uncertainty (Δ structured): *the structured singular value* μ
- for MIMO systems: $NP + RS \not\Rightarrow RP$ (not even close...)
- *RP*-conditions can not be formulated using H_∞-norms: need μ also for this