FEL3210 Multivariable Feedback Control

Lecture 6: Robust stability and performance in MIMO systems [Ch.8]

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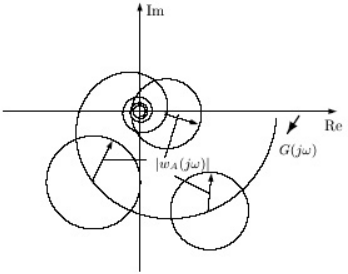
Lecture 5: model uncertainty in frequency domain

Represent uncertainty by disc at each frequency (SISO systems)

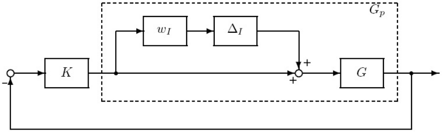


- nominal model G is center of disc
- true system assumed to lie within disc

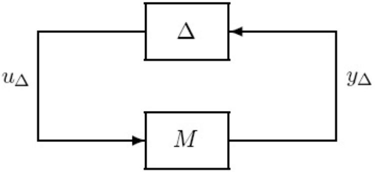
A disc with radius $|w_A(j\omega)|$ at a given frequency can be generated by

$$G_p(j\omega) = G(j\omega) + |w_A(j\omega)|\Delta_A(j\omega) \; ; \quad |\Delta_A(j\omega)| < 1$$

Lecture 5: robust stability (RS)



$\Rightarrow$



Assume $\|\Delta\|_\infty < 1$, then using *Small Gain Theorem*

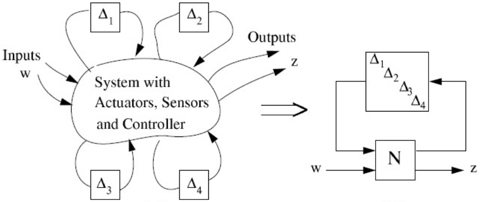
$$RS \quad \Leftrightarrow \quad \|M\|_\infty \leq 1$$

– necessary and sufficient condition for robust stability, i.e., stabilization of all plants within uncertainty set

Todays lecture

- MIMO systems: uncertainty represented by *perturbation matrices* $\Delta(s)$ which often will have a restricted *structure* (block-diagonal)
- RS problem with structured perturbation matrices can not be solved using H_∞-analysis; yields sufficient conditions only.
- Need *the structured singular value* μ to derive necessary and sufficient conditions for RS with structured $\Delta(s)$
- Robust performance (RP) problems can be cast as RS problems with structured uncertainty.

Uncertainty in MIMO systems



(a) (b)

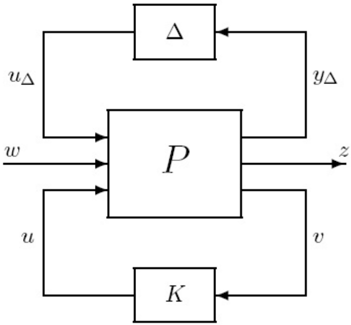
- "pull out" all sources of uncertainty into a block-diagonal matrix

$$\Delta = diag\{\Delta_i\} = \begin{pmatrix} \Delta_1 & & & \\ & \ddots & & \\ & & \Delta_i & \\ & & & \ddots \end{pmatrix}$$

  – if $\|\Delta_i\|_\infty \leq 1$ then $\|\Delta\|_\infty \leq 1$, follows from the fact that singular values of block-diagonal matrices equals singular values of blocks

General control configuration including uncertainty

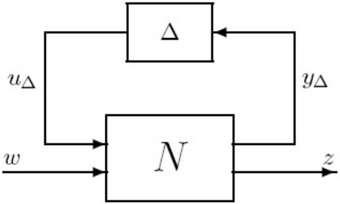
For synthesis of controller *K*



- the block-diagonal matrix $\Delta(s)$ includes all possible perturbations of the system, normed such that $\|\Delta\|_\infty \leq 1$
- performance objective: minimize the gain from *w* to *z*

Control configuration for analysis

For analysis, with given controller K,



- N is a *lower* LFT[1] of P

$$N = \mathcal{F}_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

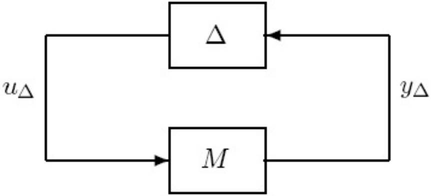
- the transfer-function $z = Fw$ is given by an *upper* LFT of N

$$F = \mathcal{F}_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$$

[1] Linear Fractional Transformation

Configuration for analysis of RS

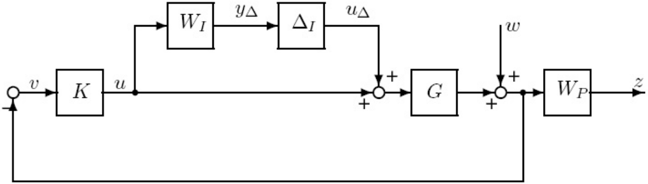
for analysis of robust stability we only need to consider



where $M = N_{11}$

- see also lecture 5

Obtaining P, N and M - an example



The generalized plant has outputs $[y_\Delta \; z \; v]$ and inputs $[u_\Delta \; w \; u]$. From block-diagram we derive

$$P = \begin{pmatrix} 0 & 0 & W_I \\ W_P G & W_P & W_P G \\ -G & -I & -G \end{pmatrix}$$

Now $N = \mathcal{F}_l(P, K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$ with

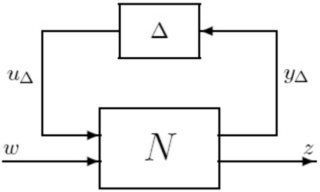
$$P_{11} = \begin{pmatrix} 0 & 0 \\ W_P G & W_P \end{pmatrix}, \; P_{12} = \begin{pmatrix} W_I \\ W_P G \end{pmatrix}, \; P_{21} = \begin{pmatrix} -G & -I \end{pmatrix}, \; P_{22} = -G$$

This yields

$$N = \begin{pmatrix} -W_I KG(I+KG)^{-1} & -W_I K(I+GK)^{-1} \\ W_P G(I+KG)^{-1} & W_P(I+GK)^{-1} \end{pmatrix}$$

And finally, $M = N_{11} = -W_I KG(I+KG)^{-1}$

Definitions of robust stability and performance



- $NS \overset{def}{\Leftrightarrow}$ N is internally stable
- $NP \overset{def}{\Leftrightarrow}$ $\|N_{22}\|_\infty < 1$
- $RS \overset{def}{\Leftrightarrow}$ $F = \mathcal{F}_u(N, \Delta)$ is stable $\forall \Delta, \|\Delta\|_\infty \leq 1$
- $RP \overset{def}{\Leftrightarrow}$ $\|F\|_\infty < 1, \forall \Delta, \|\Delta\|_\infty \leq 1$

Next: results for testing all conditions without having to search through all possible Δ's.

Generalized Nyquist criterion for RS

Assume $M(s)$ and $\Delta(s)$ stable. Then the $M-\Delta$-loop is stable if and only if det$(I - M\Delta(j\omega))$ does not encircle 0 for any Δ, any ω

$$\Leftrightarrow \quad \det(I - M\Delta) \neq 0, \ \forall\omega, \forall\Delta$$

$$\Leftrightarrow \quad \lambda_i(M\Delta) \neq 1, \ \forall i, \forall\omega, \forall\Delta$$

$$\overset{\Delta \ complex}{\Leftrightarrow} \quad |\lambda_i(M\Delta)| < 1, \ \forall i, \forall\omega, \forall\Delta$$

Thus,

$$RS \quad \Leftrightarrow \quad \rho(M\Delta) < 1, \forall\omega, \forall\Delta$$

- difficult condition to check in the general case. Must in principle consider all possible Δ's - an inifinite set.
- a sufficient condition is $\bar{\sigma}(M) < 1, \forall\omega$ (see lecture 5), but potentially highly conservative when Δ has structure

Unstructured uncertainty - Δ full matrix

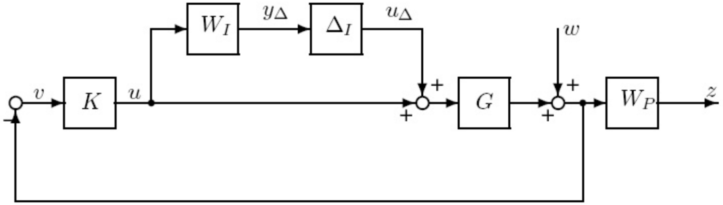
Assume Δ is a full complex matrix at each frequency. Then

$$RS \;\Leftrightarrow\; \rho(M\Delta) < 1 \;\; \forall\omega, \forall\Delta \;\Leftrightarrow\; \bar{\sigma}(M) < 1 \;\; \forall\omega \;\Leftrightarrow\; \|M\|_\infty < 1$$

Proof: we can always choose a full Δ such that $\rho(M\Delta) = \bar{\sigma}(M)$

- SVD of M: $M = U\Sigma V^H$
- choose $\Delta = VU^H$ to obtain $M\Delta = U\Sigma U^H \quad (\bar{\sigma}(\Delta) = 1)$
- $\rho(M\Delta) = \rho(U\Sigma U^H) = \rho(\Sigma) = \bar{\sigma}(M)$

Unstructured uncertainty - example

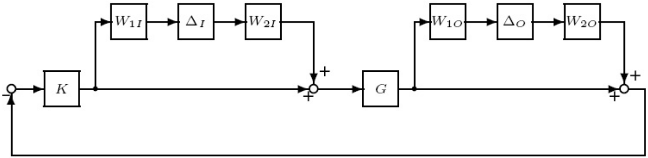


assume $\Delta_I(s)$ is a full matrix with $\|\Delta_I\|_\infty < 1$. Then

$$RS \quad \Leftrightarrow \quad \|M\|_\infty < 1 \; ; \quad M = W_I KG(1 + KG)^{-1} = W_I T_I$$

- simple condition, similar to SISO case for which $T_I = T$
- but, allowing full Δ may be highly conservative, e.g.,
  - independent input uncertainty: Δ_I diagonal
  - combining sources of uncertainty: Δ block-diagonal

Combining multiple perturbations – example



- "pull out" perturbations Δ_I and Δ_O

$$\Delta = \begin{pmatrix} \Delta_I & 0 \\ 0 & \Delta_O \end{pmatrix}$$

- from block-diagram we derive

$$M = \begin{pmatrix} -W_{I1} T_I W_{2I} & -W_{I1} K S W_{2O} \\ W_{1O} G S_I W_{2I} & W_{IO} T W_{2O} \end{pmatrix}$$

- a sufficient condition for RS is $\|M\|_\infty < 1$, but we seek a tight condition utilizing the information that Δ is structured.

Comment: lumping uncertainty into a single perturbation

- alternative to using structured Δ is to lump all uncertainties into a single perturbation, e.g., at the output
- for SISO plants, input uncertainty may be moved to the output, and vice versa, without affecting the model set Π

$$SISO: \quad G(I+\Delta_I) = (I+\Delta_O)G \quad \Rightarrow \quad \Delta_O = \Delta_I$$

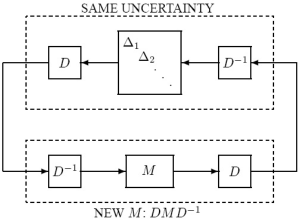
- but, for MIMO plants

$$MIMO: \quad G(I+\Delta_I) = (I+\Delta_O)G \quad \Rightarrow \quad \Delta_O = G\Delta_I G^{-1}$$

  - we get $\max_{\Delta_I} \bar{\sigma}(\Delta_O) = \bar{\sigma}(\Delta_I)\gamma(G)$, where γ is the condition number
  - diagonal Δ_I in general yields full Δ_0
- thus, be careful about moving uncertainty in MIMO systems, in particular for ill-conditioned systems.

Reducing conservatism with H_∞ and structured Δ

- with a structured Δ the condition $\bar{\sigma}(M) < 1 \ \forall\omega$ is only sufficient
- reduce conservatism by introducing scaling



where $D = diag\{d_i I_i\}$ with d_i a scalar and I_i an identity matrix of the same dimension as the block Δ_i so that

$$D\Delta = \Delta D \quad \Rightarrow \quad \Delta = D\Delta D^{-1}$$

- Then

$$RS \quad \Leftrightarrow \quad \min_{D\in\mathcal{D}} \bar{\sigma}(D(\omega)M(j\omega)D^{-1}(\omega)) < 1, \ \forall\omega$$

where $\mathcal{D}$ is the set of all matrices D such that $D\Delta = \Delta D$

The structured singular value μ

- recall generalized Nyquist criterion for $M\Delta$-structure: we seek the smallest structured Δ such that $det(I - M\Delta) = 0$
- the *structured singular value* $\mu(M)$ is defined as

$$\mu(M)^{-1} \overset{def}{=} \min_{\Delta}\{\bar{\sigma}(\Delta) | \det(I - M\Delta) = 0 \text{ for structured } \Delta\}$$

  - defined for constant complex matrices, i.e., at given frequency
  - $\mu(M)$ depends on M and structure of Δ, hence often written $\mu_\Delta(M)$

The structured singular value μ

For complex Δ

$$\mu(M) = \max_{\Delta, \bar{\sigma}(\Delta) \leq 1} \rho(M\Delta)$$

- with full Δ

$$\mu(M) = \bar{\sigma}(M)$$

  Follows since $\rho(M\Delta) \leq \bar{\sigma}(M\Delta) \leq \bar{\sigma}(M)\bar{\sigma}(\Delta)$ and we can choose $\Delta = \bar{v}\bar{u}^H$ to get $\rho(M\Delta) = \bar{\sigma}(M)$

- with repeated diagonal $\Delta = \delta I$

$$\mu(M) = \rho(M)$$

  follows since there are no degrees of freedom in the optimization problem

- in general

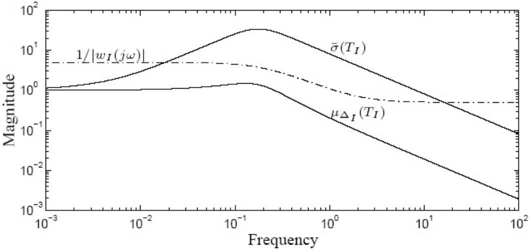
$$\rho(M) \leq \mu(M) \leq \bar{\sigma}(M)$$

Example: RS with diagonal input uncertainty

Example 8.9: decentralized PI-control of distillation process

$$G(s) = \frac{1}{\tau s + 1}\begin{pmatrix} -87.8 & 1.4 \\ -108.2 & -1.4 \end{pmatrix} \; ; \quad K(s) = \frac{\tau s + 1}{s}\begin{pmatrix} -0.0015 & 0 \\ 0 & -0.075 \end{pmatrix}$$

Input uncertainty with diagonal Δ_I and $w_I(s) = \frac{s+0.2}{0.5s+1}$



$$RS \quad \Leftrightarrow \quad \mu(w_I T_I) < 1 \; \forall\omega \quad \Leftrightarrow \quad \mu(T_I) < 1/|w_I| \; \forall\omega$$

Computing μ

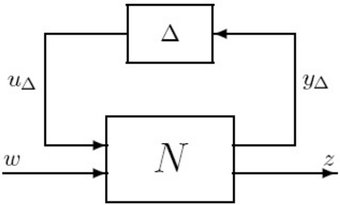
- the structured singular value μ in general not directly computable
- μ-computations based on various upper and lower bounds
- commonly used *upper bound*

$$\mu(M) \le \min_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$$

with $\mathcal{D}$ as defined above

  - convex optimization problem
  - equality applies when Δ has 3 or fewer blocks. For more blocks usually found to be a tight bound.

Robust Performance



$$\begin{pmatrix} y_\Delta \\ z \end{pmatrix} = N \begin{pmatrix} u_\Delta \\ w \end{pmatrix}$$

$$\Downarrow$$

$$z = F(N, \Delta) w$$

- assume proper scaling so that performance objective is

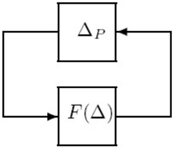
$$\max_w \frac{\|z\|_2}{\|w\|_2} < 1 \ \ \forall \omega \quad \Leftrightarrow \quad \|F(N, \Delta)\|_\infty < 1, \ \forall \Delta$$

- corresponds to robust stability condition for $M\Delta_p$-structure with $M = F(N, \Delta)$ and Δ_p a full complex perturbation!

RP cast as RS problem with structured Δ-block

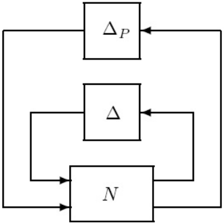
$$RP \quad \Leftrightarrow \quad \|F(\Delta)\|_\infty < 1, \forall \|\Delta\|_\infty <\leq 1$$

$$\Updownarrow$$



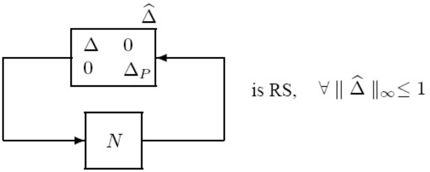
is RS, $\forall \| \Delta_P \|_\infty \leq 1$, $\forall \| \Delta \|_\infty \leq 1$

$$\Updownarrow$$



is RS, $\forall \| \Delta_P \|_\infty \leq 1$, $\forall \| \Delta \|_\infty \leq 1$

$$\Updownarrow$$

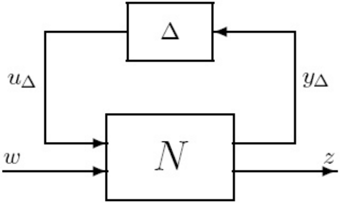


$$\Updownarrow$$

$$RP \quad \Leftrightarrow \quad \mu(N) < 1 \text{ with } \Delta = \left[\begin{pmatrix} \Delta & 0 \\ 0 & \Delta_p \end{pmatrix} \right]$$

- The robust performance problem can be formulated as a robust stability problem with structured (block-diagonal) perturbation matrix $\Delta \Rightarrow$ need μ

Summary



- NS $\Leftrightarrow$ N stable
- NP $\Leftrightarrow$ $\|N_{22}\|_\infty < 1$ & NS
- RS $\Leftrightarrow$ $\mu(N_{11}) < 1,\ \Delta = \Delta_{unc}$ & NS
- RP $\Leftrightarrow$ $\mu(N) < 1,\ \Delta = diag(\Delta_{unc}, \Delta_p)$ & NS

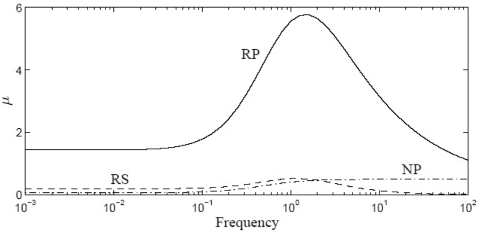
Example

8.11.3: Decoupling control of distillation column.

$$G(s) = \frac{1}{75s+1}\left[\begin{pmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{pmatrix}\right] \; ; \quad K(s) = \frac{0.7}{s} G^{-1}(s)$$

$$w_I(s) = \frac{s+0.2}{0.5s+1} \; ; \quad w_P = \frac{s/2+0.05}{s}$$

Diagonal input uncertainty



Homework: perform similar analysis for heat-exchanger from lecture 3