FEL3210 Multivariable Feedback Control

Lecture 7: Controller Synthesis and Design [Ch. 9]

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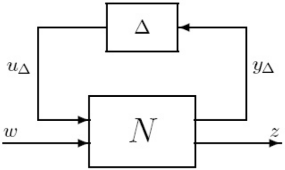
Lecture 6: Analysis of RS and RP

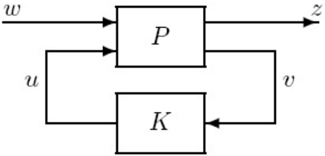
General control configuration (with given controller K):



- NS $\Leftrightarrow$ N internally stable
- NP $\Leftrightarrow$ $\|N_{22}\|_\infty < 1$ & NS
- RS $\Leftrightarrow$ $\mu(N_{11}) < 1,\ \Delta = \Delta_{unc}$ & NS
- RP $\Leftrightarrow$ $\mu(N) < 1,\ \Delta = diag(\Delta_{unc}, \Delta_p)$ & NS

Today's program: Controller synthesis

General control problem (no uncertainty):



$$z = F(P, K)w$$

Controller synthesis

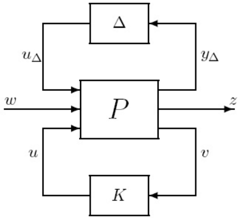
$$\min_K \|F\|_m$$

- $m = 2$: $\mathcal{H}_2$-optimal control
- $m = \infty$: $\mathcal{H}_\infty$-optimal control

Solution based on model of open-loop $P(s)$

Today's program: Controller synthesis

General control problem with uncertainty:



$$z = F(P, K, \Delta)w$$

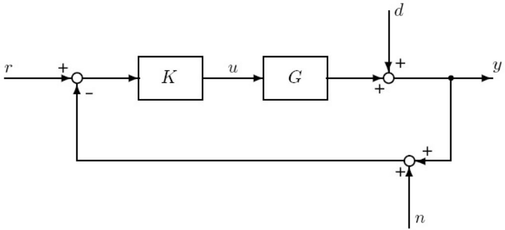
- *RS w full block uncertainty:* $\mathcal{H}_\infty$-optimal control incorporating transfer-function from u_Δ to y_Δ in an extended $F(P, K)$
- *RS w structured uncertainty & RP:* μ-synthesis

$$\min_K \max_\omega \mu_{RP}(N)$$

Todays program

- Defining $N(s) = \mathcal{F}(P(s), K(s))$ to reflect desired closed-loop properties
- (Parametrization of all stabilizing controllers) (next time)
- $\mathcal{H}_2$-optimal control
- $\mathcal{H}_\infty$-optimal control
- μ-synthesis
- The robust stabilization problem

The Control Objective



$$e = Sr - Sd + Tn$$

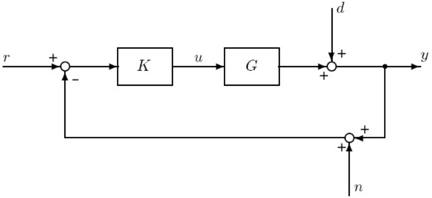
$$u = KS(r - d - n)$$

- *disturbance attenuation* and *setpoint tracking*; make S ”small”
- *noise attenuation*; make T ”small”
- *reducing input usage*; make KS ”small”
- *RS with full block input/output uncertainty*; make T_I/T ”small”

Controller design: make trade-offs between conflicting objectives

- **synthesis:** formulate and solve optimization problem
- **loop-shaping:** “manually” shape open loop-gain

Deriving $P(s)$ - signal based approach



- Minimize weighted control error e and control input u in presence of setpoint r and disturbance d

$$w = \begin{pmatrix} r \\ d \end{pmatrix} \ , \quad z = \begin{pmatrix} W_P e \\ W_u u \end{pmatrix}$$

- In open-loop (and $n = 0$)

$$z_1 = W_P e = W_P(r - d - Gu) = W_P(w_1 - w_2 - Gu) \ ; \quad z_2 = W_u u$$

$$v = e = r - d - Gu = w_1 - w_2 - Gu$$

Thus,

$$P(s) = \begin{pmatrix} W_P(s)I & -W_P(s)I & -W_P G(s) \\ 0 & 0 & W_u(s)I \\ I & -I & -G(s) \end{pmatrix}$$

- state-space realization of $P(s)$ is the basis for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis

Deriving $P(s)$ - shaping transfer-functions

Consider shaping closed-loop transfer-functions, e.g., S and T

$$\min_K \left\| \begin{pmatrix} W_P S \\ W_T T \end{pmatrix} \right\|_m \quad \Rightarrow \quad F(P,K) = \begin{pmatrix} W_P S \\ W_T T \end{pmatrix}$$

- Choose signals w and z such that

$$z = \begin{pmatrix} W_1 S \\ W_2 T \end{pmatrix} w$$

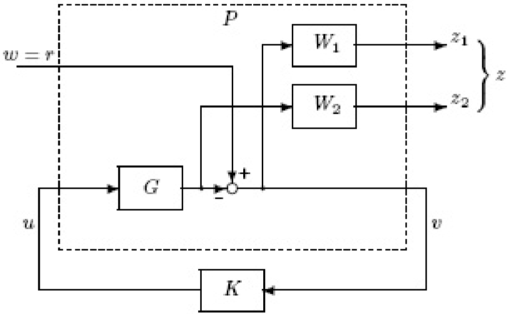
  We have $e = Sr$ and $e - r = Tr$. Thus, choose

$$w = r \; ; \quad z = \begin{pmatrix} W_1 e \\ W_2(e - r) \end{pmatrix}$$

- From this we then derive

$$P(s) = \begin{pmatrix} -W_1(s)I & -W_1(s)G(s) \\ 0 & W_2(s)G(s) \\ -I & -G(s) \end{pmatrix}$$

$P(s)$ - closed-loop shaping approach



Solving the optimization problem

Standard algorithms for solving $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal control problems are based on a state-space realization of the *generalized plant* $P(s)$

$$P = \begin{pmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix}$$

with input $[w\ u]^T$ and output $[z\ v]^T$

- Solution of optimal problem generally involves solving two Algebraic Riccati Equations (ARE)

$$A^T X + XA + XRX + Q = 0$$

- A number of assumptions on P usually need to be fulfilled to solve the optimization problem (algorithm dependent)

Some Typical Requirements on P

(A1) (A, B_2, C_2) stabilizable and detectable

- required for existence of stabilizing K

(A2) D_{12} and D_{21} have full rank

- ensures proper K

(A3) $\begin{pmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{pmatrix}$ has full column rank for all ω

(A4) $\begin{pmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{pmatrix}$ has full row rank for all ω

- ensures poles on imaginary axis detectable and controllable, respectively, in closed-loop
- avoid cancelation of poles and zeros on imaginary axis

(A5) $D_{11} = 0$ and $D_{22} = 0$

- mainly to ensure strictly proper transfer-functions (required with $\mathcal{H}_2$)

$\mathcal{H}_2$-optimal control

$$\min_K \|F(P,K)\|_2 = \min_K \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} tr[F(j\omega)F(j\omega)^H]d\omega}$$

Interpretations of $\mathcal{H}_2$-norm:

- *signal:* output covariance for white noise input

$$\|F\|_2^2 = \lim_{t\to\infty} E\{z(t)^T z(t)\}\ , \quad E\{w(t)^T w(\tau)\} = \delta(t-\tau)I$$

  – follows from Parsevals theorem

$$E\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} z(t)^T z(t)dt\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr[F(j\omega)F(j\omega)^H]d\omega = \|F\|_2^2$$

- *system:* sum of “area” of all singular values of F

$$\|F\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\sigma_i^2(F(j\omega))d\omega}$$

- minimizing output covariance for white noise input is similar to LQG!
- $\mathcal{H}_2$-optimal control problems can be solved explicitly from two algebraic Riccati equations
- Separation principle: solution can be written on the form **optimal state feedback + optimal state estimator**

LQG - a special case of $\mathcal{H}_2$-optimal control

- The LQG problem

$$\dot{x} = Ax + Bu + w_d$$
$$y = Cx + w_n$$

with

$$E\left\{ \begin{pmatrix} w_d(t) \\ w_n(t) \end{pmatrix} \begin{pmatrix} w_d(\tau)^T & w_n(\tau)^T \end{pmatrix} \right\} = \begin{pmatrix} W & 0 \\ 0 & V \end{pmatrix} \delta(t-\tau)$$

- the LQG-controller solves

$$K_{LQG} = \arg\min_K E\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T x^T Q x + u^T R u \, dt \right\}$$

with $Q = Q^T \geq 0$ and $R = R^T \geq 0$

LQG - a special case of $\mathcal{H}_2$-optimal control

- LQG cast as an $\mathcal{H}_2$-optimal control problem

$$z = \begin{pmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \; ; \quad \begin{pmatrix} w_d \\ w_n \end{pmatrix} = \begin{pmatrix} W^{1/2} & 0 \\ 0 & V^{1/2} \end{pmatrix} w$$

with $E\{w(t)^T w(\tau)\} = \delta(t-\tau)I$

Then,

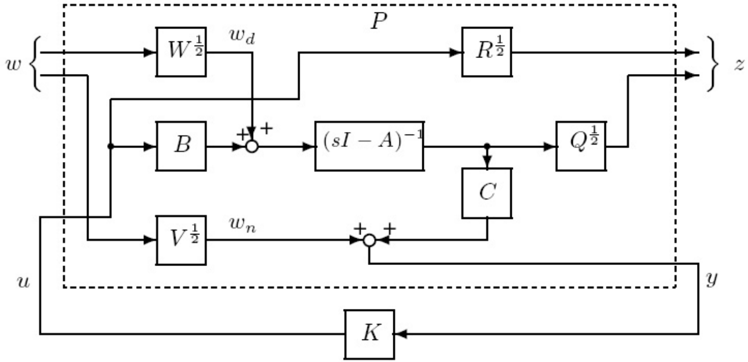
$$z = F(P,K)w \; ; \quad \|F\|_2^2 = E\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T z(t)^T z(t) dt \right\}$$

The corresponding generalized plant is

$$P = \begin{pmatrix} A & W^{1/2} & 0 & B \\ Q^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & R^{1/2} \\ C & 0 & V^{1/2} & 0 \end{pmatrix}$$

LQG cast as $\mathcal{H}_2$-optimization problem

general control configuration for LQG as $\mathcal{H}_2$-optimal control problem



The Solution

- Optimal state feedback

$$u(t) = -R^{-1}B^T X\hat{x}(t)$$

where $X = X^T \geq 0$ solves the ARE

$$A^T X + XA - XBR^{-1}B^T X + Q = 0$$

- combined with optimal state estimator

$$\dot{\hat{x}} = A\hat{x}(t) + Bu(t) + YC^T V^{-1}(y - C\hat{x})$$

where $Y = Y^T \geq 0$ solves the ARE

$$YA^T + AY - YC^T V^{-1}CY + W = 0$$

- The general $\mathcal{H}_2$-optimal controller can be separated into optimal state feedback combined with an optimal state estimator, each involving solution of an ARE

$\mathcal{H}_\infty$-optimal control

$$\min_K \|F(P,K)\|_\infty = \min_K \max_\omega \bar{\sigma}\left(F(P,K)(j\omega)\right)$$

Interpretations of $\mathcal{H}_\infty$-norm:

- *signal:* ”worst-case” amplification from input to output

$$\|F\|_\infty = \max_{w(t)\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2}$$

  ”worst-case” input is sinusoid with fixed frequency

- *system:* peak of maximum singular value

$$\|F\|_\infty = \max_\omega \bar{\sigma}(F)$$

– $\mathcal{H}_\infty$-optimal problem can in general not be solved explicitly
– but, can determine a controller that yields $\|F\|_\infty < \gamma$ for a fixed γ, if such a controller exist

the $\mathcal{H}_\infty$-optimal controller

Consider state-space realization of generalized plant

$$P = \begin{pmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix}$$

- Assume (A1)-(A5) above, and (A6) $D_{12} = \begin{pmatrix} 0 & I \end{pmatrix}^T$, $D_{21} = \begin{pmatrix} 0 & I \end{pmatrix}$, (A7) $D_{12}^T C_1 = 0$, $B_1 D_{21}^T = 0$ (A8) (A, B_1) stabilizable, (A, C_1) detectable

- Then, there exist a controller $K(s)$ such that $\|F(P,K)\|_\infty < \gamma$ if and only if the algebraic Riccati equations

$$A^T X_\infty + X_\infty A + C_1^T C_1 + X_\infty(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty = 0$$

$$A Y_\infty + Y_\infty A^T + B_1 B_1^T + Y_\infty(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y_\infty = 0$$

has solutions $X_\infty \geq 0$ and $Y_\infty \geq 0$ such that $\forall i$

$$Re\lambda_i \left[A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty\right] < 0, Re\lambda_i \left[A + Y_\infty(\gamma^{-2} C_1^T C_1 - C_2^T C_2)\right] < 0$$

$$\rho(X_\infty Y_\infty) < \gamma^2$$

– if such a solution exist then there exist a set of controllers K that satisfies $\|F(P,K)\|_\infty < \gamma$

- one specific controller, having the same number of states as $P(s)$, can be written on the form **state estimator + state feedback**

$$\dot{\hat{x}} = A\hat{x} + B_1\gamma^{-2}B_1^T X_\infty \hat{x} + B_2 u + Z_\infty L_\infty (C_2\hat{x} - y)$$

$$u = F_\infty \hat{x}$$

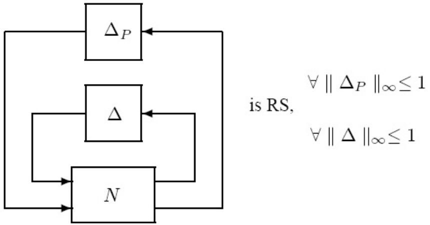
with

$$F_\infty = -B_2^T X_\infty \; ; \quad L_\infty = -Y_\infty C_2^T \; ; \quad Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}$$

- In order to determine the $\mathcal{H}_\infty$-optimal controller, iterate on γ until minimum γ for which a solution exists is found (γ-*iterations*). Convex problem.

Robustness

- $\mathcal{H}_2$-optimal control have no guaranteed robustness margins
- in $\mathcal{H}_\infty$-optimal control possible to include RS conditions with full-block uncertainty, e.g., $\|W_I T_I\|_\infty < 1$ for full block input uncertainty
- to address RS with structured uncertainty and RP: employ μ-synthesis



$$\min_K \max_\omega \mu(N(P, K))$$

μ-synthesis - $\min_K \max_\omega \mu(N)$

- no direct solution available
- employ DK-iterations based on minimizing upper bound

$$\mu(N) \leq \min_{D \in \mathcal{D}} \bar{\sigma}(DND^{-1})$$

1. **K step**: with fixed scaling D, solve $\mathcal{H}_\infty$-optimal control problem

$$\min_K \|DN(K)D^{-1}\|_\infty$$

2. **D step:** with fixed controller K, determine scaling D that minimizes scaled $\mathcal{H}_\infty$-norm

$$\min_{D \in \mathcal{D}} \|DN(K)D^{-1}\|_\infty$$

if not converged, go to 1.

– both steps convex, but no guarantee on convexity for combined problem
– controller for each step contains the number of states of $P(s)$ plus twice the number of states in $D(s)$

Robust Stabilization

Alternative to explicitly address robustness in synthesis:

1. design for performance
2. robustify design, i.e., modify controller to improve robustness (RS)

Note: addresses robust stability only.

(Left) Coprime Factorization

Consider normalized coprime factorization of $G(s)$

$$G(s) = M^{-1}(s)N(s) \quad \texttt{s.t.} \;\; M(s)M^T(-s) + N(s)N^T(-s) = I$$

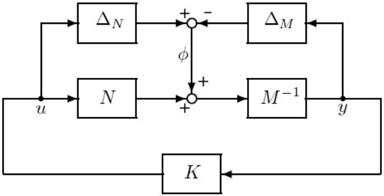
with $M(s)$ and $N(s)$ stable and coprime

- essentially $M(s)$ contains RHP poles of $G(s)$ as zeros and $N(s)$ RHP zeros as zeros
- the idea in robust stabilization is to maximize robustness wrt perturbations of the coprime factors $M(s)$ and $N(s)$

Uncertainty in Coprime Factors

Introduce uncertainty description

$$G_p(s) = (M(s) + \Delta_M(s))^{-1}(N(s) + \Delta_N(s))$$



– allows for perturbations of both poles and zeros across imaginary axis using stable perturbations $\Delta_M(s)$ and $\Delta_N(s)$

Robust Stabilization

Determine controller K that robustly stabilizes G_p with

$$\| \begin{pmatrix} \Delta_N & \Delta_M \end{pmatrix} \|_\infty \leq \epsilon$$

where ϵ is the **stability margin**

G_P may be written on $P - \Delta$-form with $\Delta = \begin{pmatrix} \Delta_N & \Delta_M \end{pmatrix}$ (full matrix!) and

$$P = \begin{pmatrix} K \\ I \end{pmatrix} (I - GK)^{-1} M^{-1}$$

thus, robust stability if $\gamma = \|P\|_\infty \leq \frac{1}{\epsilon}$

- the maximum stability margin can be explicitly computed from

$$\epsilon_{max} = \frac{1}{\gamma_{min}} = (1 - \| \begin{pmatrix} N & M \end{pmatrix} \|_H^2)^{-1/2}$$

  where $\| \cdot \|_H$ denotes the Hankel norm.

– the corresponding $\mathcal{H}_\infty$-optimal controller that yields

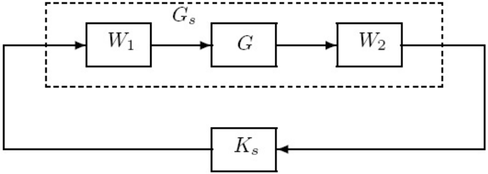
$$\left\| \left[\begin{pmatrix} K \\ I \end{pmatrix} \right] (I - GK)^{-1} M^{-1} \right\|_\infty \leq \gamma_{min}$$

can be **directly** computed by solving two algebraic Riccati equations

Glover-McFarlane Loopshaping

1. apply pre- and post-compensators to **shape loop-gain**

$$G_s(s) = W_2(s)G(s)W_1(s)$$



For instance, assume performance objective is $\|W_p S\|_\infty < 1$ and $\|W_T T\|_\infty < 1$. Then choose W_1 and W_2 to achieve

$$\omega < \omega_B : \quad \underline{\sigma}(G_s) > |W_P|$$

$$\omega > \omega_B : \quad \bar{\sigma}(G_s) < 1/|W_T|$$

2. perform **robust stabilization** of shaped plant $G_s(s)$ with K_s. If $\epsilon_{max} > 0.25$ then performance and robust stability usually not in conflict. Otherwise, if performance strongly affected by stabilization return to step 1 to modify loop-gain

Example 9.3

Consider plant with

$$G(s) = \frac{200}{10s+1} \; ; \quad G_d(s) = \frac{100}{10s+1}$$

1. choose loop-gain $|G_s| = |G_d|$, i.e.,

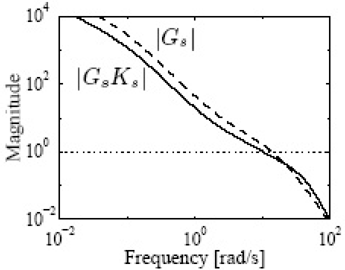
   $$|W_1| = |G^{-1}G_d| \approx 0.5$$

   add integral action, phase advance and double the gain

   $$W_1 = \frac{s+2}{s}$$

   gives oscillatory response
2. maximum stability margin: use e.g., *ncfsyn* in Matlab to find $\gamma_{min} = 2.34$, or $\epsilon_{max} = 0.43$
3. choose $\gamma = 1.1\gamma_{min}$ and compute corresponding robustifying controller K_s.

Example 9.3: effect of robustification



Next Time

- Parametrization of all stabilizing controllers
- LMI formulation of $\mathcal{H}_2 - /\mathcal{H}_\infty$-optimal control problems
- Model reduction (brief introduction)
- Control structure design (brief introduction)
- Course summary