FEL3210 Multivariable Feedback Control

Lecture 8: Youla parametrization, LMIs, Model Reduction and Summary [Ch. 11-12]

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Todays program

Optimal control problems

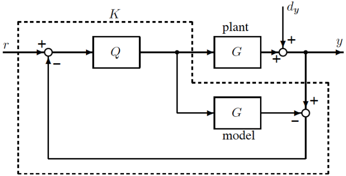
$$\min_K \|N(K,P)\|_m, \;\; m = 2, \infty$$

- **Youla parametrization:** search over all stabilizing $K(s) \Rightarrow$ search over all stable transfer-functions $Q(s)$
  - the *model matching problem*
- **Linear Matrix Inequalities:** translate optimization problem into low-complexity convex problem
- **Model reduction:** optimization problem typically yields high-order $K(s) \rightarrow$ reduce order of controller while maintaining essential properties

- Learning outcome?
- Information about exam.

Paramterization of all stabilizing controllers

Internal Model Control (IMC) structure



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- Assume G stable. Then closed-loop internally stable iff

$$K(I+GK)^{-1} = Q \qquad (I+GK)^{-1} = I - GQ$$
$$(I+KG)^{-1} = I - QG \quad G(I+KG)^{-1} = G(I - QG)$$

  all stable $\Leftrightarrow$ Q stable
- The error feedback controller $K = Q(I - GQ)^{-1}$
- Thus, a parametrization of all stabilizing controllers is

$$K = Q(I - GQ)^{-1}$$

  where $Q(s)$ is any stable transfer-function matrix

Example: H_∞ Model Matching Problem

For stable G, find controller such that

$$\|w_P S\|_\infty < 1$$

- From IMC

$$S = I - GQ$$

- Introduce $P_1 = w_P$, $P_2 = w_P G$, $P_3 = I$, $\gamma = 1$. Then

$$\|w_P S\|_\infty < 1 \quad \Leftrightarrow \quad \|P_1(s) - P_2(s)Q(s)P_3(s)\|_\infty < \gamma$$

- Known as the *model matching problem*

Parametrization for unstable plants

- Left coprime factorization

$$G(s) = M^{-1}(s)N(s)$$

such that M, N proper, stable and satisfy *Bezout identity*

$$NX + MY = I$$

- Parametrization of all stabilizing controllers

$$K(s) = (Y(s) - Q(s)N(s))^{-1}(X(s) + Q(s)M(s))$$

where $Q(s)$ is any stable transfer-matrix

Todays program

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Linear Matrix Inequalities

Linear Matrix Inequality (LMI)

$$F_0 + F_1 x_1 + F_2 x_2 + \dots F_m x_m > 0$$

where

- $x = [x_1 \dots x_m]$ is a real vector
- $F_i, i = 0, m$ are symmetric real matrices

An LMI imposes a *convex constraint* on x

- **feasibility problem:** find some x that satisfies LMI
- **optimization problem:** minimize $c^T x$ subject to LMI

Old problem, efficient solvers now available

Remark: LMIs often written on matrix form also for x

$$F_0 + \Sigma_{i=1}^{n} G_i X_i H_i > 0$$

where G_i, H_i are given matrices and we seek X_i

Example 1: Linear stability problem

LTI system

$$\dot{x} = Ax(t)$$

- Lyapunov function $V(x) = x^T P x > 0$ with $\dot{V}(x) < 0$ iff

$$P = P^T > 0, \quad A^T P + PA < 0$$

- Corresponds to an LMI feasibility problem (just stack the two inequalities into one big matrix)

Example 2: Linear robust stability problem

Polytopic LTV system

$$\dot{x} = A(t)x(t)\ , \quad A(t) \in \{A_1, \ldots, A_L\}$$

- Lyapunov function exist iff

$$P = P^T > 0\ , \quad A_i^T P + PA_i < 0,\ i = 1, \ldots L$$

- LMI feasibility problem

Optimization problems: $\mathcal{H}_\infty$-norm

Consider LTI system

$$\begin{aligned} \dot{x} &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) + Dw(t) \end{aligned}$$

The $\mathcal{H}_\infty$ norm of G_{zw} is equivalent to solving

$$\min \gamma \quad s.t. \quad \begin{pmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} < 0, \quad P > 0$$

i.e., minimization subject to LMI.

- Computing upper bound on structured singular value μ via $\min_D \bar{\sigma}(D^{-1} N D)$ is another example that can be cast as an optimization problem with LMI constraint.
- Solutions to Algebraic Riccati equations, e.g., in $\mathcal{H}_2/\mathcal{H}_\infty$-optimal control, can be obtained via LMI feasibility problem.

LMIs in control - the Essence

- Many problems in optimal and robust control can be cast as LMI problems $\Rightarrow$ convex optimization problems for which efficient algorithms exist (e.g., interior point methods)
- Matlab: LMI toolbox, and LMI Lab in Robust Control toolbox.
- See *EL3300 Convex Optimization with Engineering Applications*

Todays program

Optimal control problems

$$\min_K \|N(K,P)\|_m,\ m = 2, \infty$$

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The LTI Model Reduction Problem

Given minimal state-space model (A, B, C, D)

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) & \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y(t) &= Cx(t) + Du(t) & \qquad y \in \mathbb{R}^l \end{aligned}$$

or, as input-output model

$$Y(s) = \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{G(s)} U(s)$$

- **model reduction:** we seek a model

$$\begin{aligned} \dot{x}_r &= A_r\hat{x}(t) + B_r u(t) & \qquad x_r \in \mathbb{R}^k, u \in \mathbb{R}^m \\ y(t) &= C_r x_r(t) + D_r u(t) & \qquad y \in \mathbb{R}^l \end{aligned}$$

$$G_a(s) = C_r(sI - A_r)^{-1}B_r + D_r$$

with $k < n$, such that the predicted input-output behavior is close in some sense, e.g., $\|G - G_a\|_\infty$ is small

Why Model Reduction?

- Reduced computational complexity
  - time for dynamic simulation is approximately proportional to n^3 (if A dense)
  - in particular, important for real time applications, e.g., controllers
- Controller synthesis methods typically yield controllers that have order at least equal to model order, usually significantly higher. Thus, to obtain low order controller
  - reduce model order prior to control design, or
  - reduce controller order after design

A multitude of model reduction methods. Here we will consider those most commonly employed in linear control theory.

Truncation and Residualization

Divide state vector x into two vectors x_1 and x_2 of dimension k and $n-k$, respectively

$$\begin{aligned}\dot{x}_1 &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t)\\ \dot{x}_2 &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t)\\ y(t) &= C_1x_1(t) + C_2x_2(t) + Du(t)\end{aligned}$$

We aim at removing the state vector x_2, i.e., obtain a kth order model from an nth order model

- **Truncation:** let $x_2 = 0$, i.e., remove x_2 from state-space model
- **Residualization:** let $\dot{x}_2 = 0$, i.e., x_2 becomes an algebraic variable which depends on x_1 and u

Truncation

With $x_2 = 0$ we get

$$(A_r, B_r, C_r, D_r) = (A_{11}, B_1, C_1, D)$$

- Simply removing a number of states makes little sense in general
- Consider first transforming (A, B, C, D) into *Jordan form* and arrange the states so that x_2 correspond to the fastest modes
- If the Jordan form is diagonal (distinct eigenvalues λ_i) then

$$G(s) = \sum_{i=1}^{n} \frac{c_i b_i^T}{s - \lambda_i}$$

- Removing the $n - k$ fastest modes then yields the model error

$$G(s) - G_a(s) = \sum_{i=k+1}^{n} \frac{c_i b_i^T}{s - \lambda_i} \quad \Rightarrow \quad \|G - G_a\|_\infty \leq \sum_{i=k+1}^{n} \frac{\bar{\sigma}(c_i b_i^T)}{|Re(\lambda_i)|}$$

note: must assume stable $G(s)$

Truncation cont'd

- The H_∞ error bound
$$\sum_{i=k+1}^{n} \frac{\bar{\sigma}(c_i b_i^T)}{|Re(\lambda_i)|}$$
depends not only on eigenvalues of fast modes, but also on the residues $c_i b_i^T$, i.e., the effect of inputs u on x_2 and effect of x_2 on outputs y
- At $\omega = \infty$
$$G_a(i\infty) = G(i\infty) = D$$
Thus, no error at infinite frequency

Residualization

With $\dot{x}_2 = 0$ we get (assume A_{22} invertible)

$$x_2(t) = -A_{22}^{-1} A_{21} x_1(t) - A_{22}^{-1} B_2 u(t)$$

and elimination of x_2 from partitioned model then yields

$$\dot{x}_1(t) = (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1(t) + (B_1 - A_{12} A_{22}^{-1} B_2) u(t)$$

$$y(t) = (C_1 - C_2 A_{22}^{-1} A_{21}) x_1(t) + (D - C_2 A_{22}^{-1} B_2) u(t)$$

- Thus, the reduced model $(A_r, B_r, C_r, D_r) =$

  $$(A_{11} - A_{12} A_{22}^{-1} A_{21}, B_1 - A_{12} A_{22}^{-1} B_2, C_1 - C_2 A_{22}^{-1} A_{21}, D - C_2 A_{22}^{-1} B_2)$$

- Corresponds to a *singular perturbation method* if A transformed to Jordan form first
- At zero frequency

  $$G_a(0) = G(0)$$

  follows from the fact that $\dot{x}_2 \equiv 0$ at steady-state

Comments on Truncation and Residualization

- Truncation gives best approximation at high frequencies
- Residualization gives best approximation at low frequencies
- The two methods are related through the bilinear transformation $s \to \frac{1}{s}$
- Both methods can in principle give rise to arbitrarily large model reduction errors since effect of states on input-output behavior not only related to the speed of response
- Should be combined with some method that ensures relatively small overall effect of removed states on input-output behavior $\Rightarrow$ *balancing*

The Controllability Gramian

- The state space model (A, B, C, D) has an impulse response from $u(t)$ to $x(t)$ given by

$$X(t) = e^{At}B$$

- A quantification of the "size" of the impulse response is

$$P(t) = \int_0^t X(\tau)X^T(\tau)d\tau = \int_0^t e^{A\tau}BBe^{A^T\tau}d\tau$$

- Define the *Controllability Gramian P* as

$$P = \lim_{t\to\infty} P(t)$$

- The controllability gramian can be computed from the Lyapunov equation

$$AP + PA^T + BB^T = 0$$

- P is a quantitative measure for controllability of the different states. Essentially, measures the effect of the inputs on the different states

The Observability Gramian

- The state space model (A, B, C, D) with input $u(t) = 0$ and initial state $x(0) = x^*$ has the output

$$y(t) = Ce^{At}x^*$$

- The energy of the output

$$\int_0^t y^T(\tau)y(\tau)d\tau = x^{T*}\underbrace{\int_0^t e^{A^T\tau}C^TCe^{A\tau}d\tau}_{Q(t)}\,x^*$$

- Define the *Observability Gramian* Q as

$$Q = \lim_{t\to\infty}\int_0^t e^{A^T\tau}C^TCe^{A\tau}d\tau$$

- The observability gramian can be computed from the Lyapunov equation

$$A^TQ + QA + C^TC = 0$$

- Q is a quantitative measure for observability of the different states. Essentially, measures the effect of states on outputs

Balanced Realizations

- We seek a similarity transformation of the states $x_b(t) = Tx(t)$ so that the transformed state space model

$$\dot{x}_b = TAT^{-1}x_b(t) + TBu(t)$$
$$y(t) = CT^{-1}x_b(t) + Du(t)$$

  has controllability and observability gramians

$$P = Q = diag(\sigma_1, \ldots, \sigma_n); \quad \sigma_i = \sqrt{\lambda_i(PQ)}$$

  where the *Hankel singular values* $\sigma_1 > \sigma_2 > \ldots > \sigma_n$

- Each state x_{bi} in the balanced realization is as observable as it is controllable, and σ_i is a measure of how controllable/observable it is
- A state with a relatively small σ_i has a relatively small effect on the input-output behavior and can hence be removed without significantly affecting the predicted input-output behavior

Balanced Truncation and Residualization

- Consider the *balanced realization* (A, B, C, D) of $G(s)$ with partitioning

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

$$P = Q = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$$

  where $\Sigma_1 = diag(\sigma_1, \ldots, \sigma_k)$ and $\Sigma_2 = diag(\sigma_{k+1}, \ldots, \sigma_n)$

- A balanced truncation or residualization retaining the k states corresponding to Σ_1 will both have model reduction error

$$\|G - G_a^k\|_\infty \leq 2 \sum_{i=k+1}^{n} \sigma_i$$

  "twice the sum of the tail"

- May in principle include frequency dependent weighting to emphasize certain frequency ranges, However, this introduces extra states and it is furthermore usually non-trivial to choose weigths that give the desired result

Optimal Hankel Norm Approximation

- *The Hankel norm* of a transfer-matrix $E(s)$

$$\|E(s)\|_H = \sigma_1 = \sqrt{\rho(PQ)}$$

  i.e., equals the maximum Hankel singular value of $E(s)$

- Optimal Hankel norm approximations seeks to minimize $\|G - G_a^k\|_H$ for a given order k of the reduced order model
- For stable square $G(s)$ the optimal Hankel norm kth order approximation can be directly computed and has Hankel norm error

$$\|G - G_a^k\|_H = \sigma_{k+1}$$

- The optimal Hankel norm is independent of the D-matrix of G_a^k The minimum ∞-norm of the error is

$$\min_D \|G - G_a^k\|_\infty \leq \sum_{i=k+1}^{n} \sigma_i$$

  i.e., "sum of the tails" only

Unstable models

Balanced truncation and residualization and optimal Hankel norm approximations applies to stable $G(s)$ only. Two “tricks” to deal with unstable models

1. Separate out unstable part before performing model reduction of stable part
$$G(s) = G_u(s) + G_s(s) \quad \Rightarrow \quad G_a(s) = G_u(s) + G_{sa}(s)$$
2. Consider coprime factorization of $G(s)$
$$G(s) = M^{-1}(s)N(s)$$
with $M(s)$ and $N(s)$ stable. Apply model reduction to $[M(s)\ N(s)]$ and use
$$G_a(s) = M_a^{-1}(s)N_a(s)$$

Model Reduction in Matlab

- `modreal`: truncation or residualization
- `slowfast`: slow/fast mode decomposition
- `balreal`: balanced realization
- `hankelmr`: optimal Hankel norm approximation
- `stabproj`: decompose into stable and antistable parts
- `ncfmr`: balanced model truncation for normalized coprime factors

Homework 8: test it out yourself! (no hand in).

- use e.g., `rss(n)` to generate a random stable state-space model with n states.
- compare step and frequency responses of different reduced models

Model Reduction Course

- EL3500 Introduction to Model Order Reduction

Learning Outcomes

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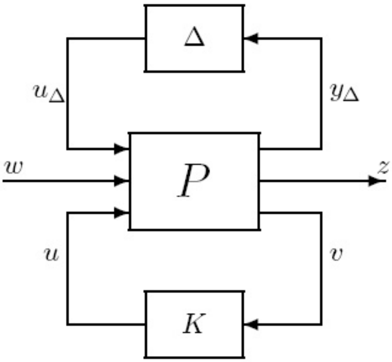
Learning Outcomes



The Final Moment

Exam:

- Covers Lectures and Ch. 1-9 (+ Ch. 11-12 tutorial) in Skogestad and Postlethwaite
- 1-day take home exam, open book.
  - allowed aids: course book(s), lecture slides, matlab, calculator
  - not allowed: old exams, exercises, solutions
- available between March 29 - April 15
- send an email to jacobsen@kth.se with the date at which you want to take it (give at least 2 days notice)
- All exercises must be approved prior to taking out exam