Introduction to linear matrix inequalities

Wojciech Paszke

Institute of Control
and Computation Engineering,
University of Zielona Góra, Poland
e-mail: W.Paszke@issi.uz.zgora.pl

Outline

Mile-steps publications

- S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *SIAM studies in applied mathematics*, SIAM, Philadelphia, 1994. Available at: `https://web.stanford.edu/~boyd/lmibook/lmibook.pdf`.
- P. Gahinet, A. Nemirovski, A. J. Laub and M. Chilali, *LMI Matlab Control Toolbox*, The MathWorks Inc., The Mathworks Partner Series, 1995.
- Y. Ebihara, D. Peaucelle, D. Arzelier, *S-Variable Approach to LMI-Based Robust Control*, Communications and Control Engineering, Springer-Verlag, London 2015.
- J. G. VanAntwerp, R. D. Braatz, *A tutorial on linear and bilinear matrix inequalities*, Journal of Process Control, vol. 10, 2000, pp. 363-385.
- and many, many others by Gahinet, Henrion, Bachelier, Chilali, Boyd, Balakrishnan and others.

Introduction to LMIs

Available solvers

- **Matlab LMI Control Toolbox** - since MATLAB 7 the LMI CONTROL TOOLBOX is incorporated into the Robust Control Toolbox (details available at `www.mathworks.com`).
- **Scilab LMITool** available at `www.scilab.org`
- **SeDuMi** (download it from `sedumi.ie.lehigh.edu/downloads`) (+ **YALMIP** as a parser - available at `users.isy.liu.se/johanl/yalmip/`)
- others (plenty of those released lately, indeed any SDP solver can be used to solve LMI)

Properties of positive definite matrices

Definition

The matrix P is a positive definite matrix iff P is symmetric, $\lambda_i(P) \in \mathbb{R}$ and $\lambda_{min}(P) > 0$.
$P > 0$ means that P is positive definite.

Properties

- $P > 0 \Leftrightarrow x^T P x > 0 \ \forall x \neq 0$
- $svd(P) = U \Lambda U^T$ where U is a unitary matrix ($UU^T = I$)
- $P > 0 \Leftrightarrow P^{-1} > 0$ and hence P is non-singular.
- if $A > 0$ and $B > 0$ then $A + B > 0$
- if $A > 0$ and $B > 0$ then

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} > 0$$

- $P > 0 \Leftrightarrow -P < 0$

Linear System Stability

Continuous-time systems

Lyapunov methods

The Lyapunov method for analyzing stability is described in most texts on process and system dynamics. The basic idea is to search for a positive definite function of the state (called the Lyapunov function - in the simplest form $V(x) = x(t)^T P x(t)$ where P is a symmetric positive definite matrix) whose time derivative (i.e. increment) $\frac{d}{dt}V(t)$ is negative definite.

Interpretation

- $V(t)$ has meaning of process energy,
- $\frac{d}{dt}V(x) < 0$ means that $\lim_{t\to\infty} x(t) = 0$ i.e. the system is stable.

Linear System Stability

Continuous-time systems

Continuous-Time (CT) Linear Time-Invariant (LTI) System

$$\dot{x}(t) = Ax(t), \quad x(0) \neq 0, \quad x(t) \in \mathbb{R}^n \tag{1}$$

The CTLTI system (1) is said to be **asymptotically stable** if

$$\lim_{t \to \infty} x(t) = 0, \quad \forall x(0) \neq 0$$

The CTLTI system (1) is said to be stable **in the sense of Lyapunov** if there exists a Lyapunov function $V(x)$ such that

$$\frac{d}{dt} V(x) < 0;$$

Linear System Stability

Continuous-Time systems

Since $V(x) = x(t)^T P x(t)$ and $P > 0$ then

$$\frac{d}{dt}V(x) = \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t)$$

Taking into account (1) we obtain

$$x(t)^T (A^T P + PA) x(t) < 0, \forall x(t) \neq 0$$

and hence

$$A^T P + PA < 0$$

This is an LMI, where P is the matrix variable.

Linear System Stability

Discrete-time systems

Discrete-Time (DT) Linear Time-Invariant (LTI) System

$$x_{k+1} = Ax_k, \quad x_0 \neq 0, \quad x_k \in \mathbb{R}^n \tag{2}$$

The DTLTI system (2) is said to be **asymptotically stable** if

$$\lim_{k \to \infty} x_k = 0, \quad \forall x_0 \neq 0$$

The DTLTI system (2) is said to be stable **in the sense of Lyapunov** if there exists a Lyapunov function $V(x)$ such that

$$V(x_{k+1}) - V(x_k) < 0;$$

Linear System Stability

Discrete-time systems

Stability

The following statements are equivalent:

- The system (2) is asymptotically stable.
- There exists a quadratic Lyapunov function

$$V(x) := x^T P x > 0, \quad P \in \mathbb{S}^n$$

  such that the system (2) is stable in the sense of Lyapunov.
- $\max_i \|\lambda_i(A)\| < 1$.

Linear System Stability

Lyapunov Stability Test

Given the system (2) find if there exists a matrix $P \in \mathbb{S}^n$ such that

a) $V(x) := x^T P x > 0; \forall x \neq 0$

b) $V(x_{k+1}) - V(x_k) < 0; \;\; \forall x_{k+1} = A x_k, \;\; x \neq 0$

Remarks

$$V(x) := x^T P x > 0, \; \forall x \neq 0 \Rightarrow \quad P > 0$$

$$V(x_{k+1}) = x_{k+1}^T P x_{k+1} = x_k^T A^T P A x_k$$

therefore

$$V(x_{k+1}) - V(x_k) = x_k^T A^T P A x_k - x_k^T P x_k$$

$$V(x_{k+1}) - V(x_k) = x_k^T \left(A^T P A - P \right) x_k$$

finally

$V(x_{k+1}) - V(x_k) < 0$ if, and only if $A^T P A - P < 0$

Linear System Stability

Lyapunov Stability Test

Given the system (2) find if there exists a matrix $P \in \mathbb{S}^n$ such that the **LMI** (Linear Matrix Inequality)

$$P > 0, \quad A^T PA - P < 0$$

is **feasible**.

Note that there exist methods which allow us to solve the stability problem by direct and more effective methods, e.g.

- compute the eigenvalues of the matrix A
- solve the Lyapunov equality

$$P > 0, \quad A^T PA - P = -I$$

Yalmip code for stability test

```
yalmip('clear')
% Data
A=[0 1 0; 0 0 1;-0.2 0.3 -0.1];
n=size(A,1)
P=sdpvar(n,n,'symmetric'); % matrix variable
F=set(P>0) % P has to be positive definite
F=F+set(A'*P*A-P< 0) % LMI for stability
Sol=solvesdp(F) % solution
P=double(P) % numerical result
eig(P)  % check if OK
checkset(F)
```

Linear Matrix Inequalities

Formal definition

A **linear matrix inequality** (LMI) is an expression of the form (the canonical form)

$$\mathbf{F}(\mathbf{x}) := \mathbf{F}_0 + x_1\mathbf{F}_1 + \cdots + x_n\mathbf{F}_n < 0$$

where

- $\mathbf{x} = (x_1, \ldots, x_n)$ - vector of unknown scalar entries (decision variables)
- $\mathbf{F_0}, \ldots, \mathbf{F_n}$ - known symmetric matrices
- $< \mathbf{0}$' - negative definiteness (in many publications $\prec 0$ for the matrix notation is used instead of < 0 - depends on you!)

Features of LMIs

LMIs have several intrinsic and attractive features

1. An LMI is convex constraint on x (a convex feasibility set). That is, the set $\mathcal{S} := \{x : F(x) < 0\}$ is convex. Indeed, if $x_1, x_2 \in \mathcal{S}$ and $\alpha \in (0, 1)$ then

$$F(\alpha x_1 + (1-\alpha)x_2) = \alpha F(x_1) + (1-\alpha)F(x_2) > 0$$

   where in the equality we used that F is affine and the inequality follows from the fact that $\alpha \geqslant 0$ and $(1-\alpha) \geqslant 0$.

2. While the constraint is matrix inequality instead of a set of scalar inequalities like in linear programming (LP), a much wider class of feasibility sets can be considered.

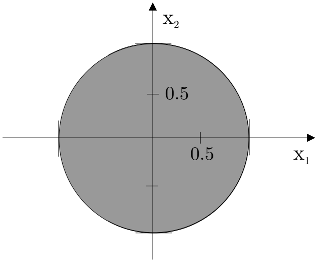
3. Thirdly, the convex problems involving LMIs can be solved with powerful interior-point methods. In this case "*solved*" means that we can find the vector of the decision variables x that satisfies the LMI, or determine that *no solution* exists.

Example - part 1

To confirm that the feasibility set represented by LMI is the convex set, the following inequality is now considered

$$\underbrace{\begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ x_1 & x_2 & 1 \end{bmatrix}}_{F(x)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{F_0} + x_1 \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{F_1} + x_2 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{F_2} > 0$$

In this case, we see that the feasible set is the interior of the unit disc $\left(\sqrt{x_1^2 + x_2^2} \leqslant 1\right)$,



Example - part 2

Note that the **Schur complement** (details will be given) of the block

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

in previous example gives the equivalent condition

$$1 - \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0 \quad \Leftrightarrow \quad 1 - (x_1^2 + x_2^2) > 0$$

Example

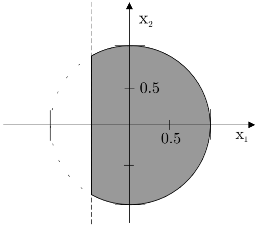
Two convex sets described by the previous example and

$$x_1 + 0.5 > 0$$

are given. They can be expressed by the LMI of the form

$$\left[\begin{array}{ccc|c} 1 & 0 & x_1 & 0 \\ 0 & 1 & x_2 & 0 \\ x_1 & x_2 & 1 & 0 \\ \hline 0 & 0 & 0 & x_1 + 0.5 \end{array}\right] > 0$$

which represents the convex set depicted below (the intersection of hyperplane and the interior of the unit circle).



Linear Matrix Inequalities cont.

Consider the stability condition for CTLTI system described by $\dot{x}(t) = Ax(t)$, which states that the system is stable iff there exists a matrix $P > 0$ such that the LMI of $A^T P + PA < 0$ is satisfied. Assume that $n = 2$. To present it in the canonical form note that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; P = P^T = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

Next, the multiplication due to the structures of matrices A and P provides the following inequality (note that since a_{ij} and p_{kl} are scalars then $a_{ij}p_{kl} = p_{kl}a_{ij}$)

$$\begin{bmatrix} 2p_{11}a_{11} + 2p_{12}a_{21} & p_{12}(a_{11} + a_{22}) + p_{22}a_{21} + p_{11}a_{12} \\ p_{12}(a_{11} + a_{22}) + p_{22}a_{21} + p_{11}a_{12} & 2p_{12}a_{12} + 2p_{22}a_{22} \end{bmatrix} < 0$$

or write it in the canonical form as

$$p_{11}\begin{bmatrix} 2a_{11} & a_{12} \\ a_{12} & 0 \end{bmatrix} + p_{12}\begin{bmatrix} 2a_{21} & a_{22} + a_{11} \\ a_{22} + a_{11} & 2a_{12} \end{bmatrix} + p_{22}\begin{bmatrix} 0 & a_{21} \\ a_{21} & 2a_{22} \end{bmatrix} < 0$$

Numerical Solution: Interior-point Algorithm

- Basic idea
  - Construct a barrier function $\phi(x)$ that is well defined for strict feasible x and is $-\varepsilon$ (where $-\infty < \varepsilon \ll 0$) only at the optimal $x = x^*$ e.g.

$$\phi(x) = -\log\det(F(x)) = \log\det\left(F^{-1}(x)\right)$$

  - Generate a sequence $\{x^{(k)}\}$ so that

$$\lim_{k\to\infty} \phi(x^{(k)}) = -\gamma$$

  - Stop if $\phi(x^{(k)})$ is negative enough

- polynomial-time algorithm - number of flops bounded by $\mathbf{mn^3}\log(\mathbf{C}/\epsilon)$ (for accuracy $< \epsilon$)

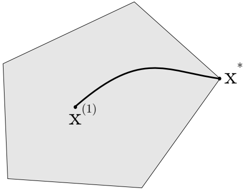
where m is row size of the LMI, n denotes number of decision variables and C is a scaling factor.

  - unconstrained optimization problem

$$\begin{aligned}\min f(x) &= \min f_0(x) + \mu\phi(x)\\ &= c^T x - \mu\log\det(F(x))\end{aligned}$$

  - Application of the Newton-like method

$$H_k \Delta x_k = -t_k$$



Analytic solution of the LMI problem

It can be shown that the LMI is equivalent to n polynomial inequalities. To see consider the well-known result in matrix theory is positive definite iff, all of its principal minors $m_i(x)$ are positive. This means that the principal minors are multivariate polynomials of indeterminates x_i i.e.

$$
\begin{aligned}
m_1(x) =& F(x)_{11} = F_{0_{11}} + \sum_{i=1}^{n} x_i F_{i_{11}} \\
m_2(x) =& \det\left(\begin{bmatrix} F(x)_{11} & F(x)_{12} \\ F(x)_{21} & F(x)_{22} \end{bmatrix}\right) = \left(F_{0_{11}} + \sum_{i=1}^{n} x_i F_{i_{11}}\right)\left(F_{0_{22}} + \sum_{i=1}^{n} x_i F_{i_{22}}\right) \\
& - \left(F_{0_{21}} + \sum_{i=1}^{n} x_i F_{i_{21}}\right)\left(F_{0_{12}} + \sum_{i=1}^{n} x_i F_{i_{12}}\right)
\end{aligned}
$$

Analytic solution of the LMI problem - cont.

$$
\begin{aligned}
m_k(x) =& \det\left(\begin{bmatrix} F(x)_{11} & \cdots & F(x)_{1k} \\ \vdots & \ddots & \vdots \\ F(x)_{k1} & \cdots & F(x)_{kk} \end{bmatrix}\right) \\
m_n(x) =& \det(F(x)) = \det\left(\begin{bmatrix} F(x)_{11} & \cdots & F(x)_{1n} \\ \vdots & \ddots & \vdots \\ F(x)_{n1} & \cdots & F(x)_{kn} \end{bmatrix}\right)
\end{aligned}
$$

where $F(x)_{kl}$ denotes the element on k-th row and l-th column of $F(x)$.

Analytic solution of the LMI problem - an example

Consider again the problem of finding a block-diagonal matrix $P > 0$ ($P = \mathrm{diag}\,(P_h, P_v)$) such that the following LMI

$$A^T PA - P < 0$$

or

$$-A^T PA + P > 0 \tag{3}$$

is satisfied. Since $P = \mathrm{diag}(x_1, x_2)$ and the matrix A is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0.4942 & 0.5706 \\ 0.1586 & 0.4662 \end{bmatrix}$$

Analytic solution of the LMI problem - an example

The solution of the LMI (3) is equivalent to the solution of the set of inequalities

$$
\begin{aligned}
m_1(x) &= -x_1(a_{11}^2-1) - x_2 a_{21}^2 = 0.75576636x_1 - 0.02515396x_2 > 0 \\
m_2(x) &= -x_1(a_{11}^2-1) - x_2 a_{21}^2 = 0.32558436x_1 + 0.78265756x_2 > 0 \\
m_3(x) &= \left(-x_1(a_{11}^2-1) - x_2 a_{21}^2\right)\left(-x_1 a_{12}^2 - x_2(a_{22}^2-1)\right) \\
&\quad -(-x_1 a_{12} a_{11} - x_2 a_{22} a_{21})(-x_1 a_{12} a_{11} - x_2 a_{22} a_{21}) \\
&= -0.32558436x_1^2 + 0.5579956166x_1x_2 - 0.02515396x_2^2 > 0
\end{aligned} \tag{4}
$$

with

$$x_1 > 0 \text{ and } x_2 > 0 \tag{5}$$

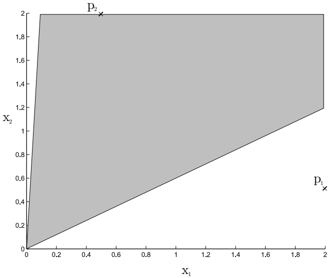
On the other hand, recall that the LMI is a convex set in $\mathbb{R}^n$ defined as

$$\mathcal{F} = \left\{ x \in \mathbb{R}^n : F(x) = F_0 + \sum_{i=1}^{n} x_i F_i > 0 \right\}$$

which can be described in terms of principal minors as

$$\mathcal{F} = \{ x \in \mathbb{R}^n : m_i(x) \geqslant 0,\ i = 1, \ldots, n \}$$

Hence the inequalities (4) and (5) describe the convex set



Final check

To validate the result, computations for two points $p_1 = (x_1, x_2) = (2, 0.5)$ and $p_2 = (x_1, x_2) = (0.5, 2)$ will be provided. First consider the point p_1. In this case, the matrix below is obtained

$$R = A^T PA - P = \left[\begin{array}{cc} -1.4990 & 0.6010 \\ 0.6010 & 0.2598 \end{array} \right]$$

Because eigenvalues of the matrix R are $\lambda_1 = -1.6847$ and $\lambda_2 = 0.4456$, it is clear that p_1 is not the solution of the considered LMI (see that p_1 does not lie inside the feasible set). Taking p_2 into computation yields

$$R = \left[\begin{array}{cc} -0.3276 & 0.2889 \\ 0.2889 & -1.4025 \end{array} \right]$$

which is negative defined (its eigenvalues are $\lambda_1 = -1.4752$ and $\lambda_2 = -0.2549$).

On the other hand, evaluating the principal minors (4) yields

Table: Principal minors.

	p_1	p_2
$m_1(x)$	0.3759836866	0.3759836866
$m_2(x)$	-0.259839940	1.402522940
$m_3(x)$	1.498955740	0.327575260

These results clearly show that in the case of the point (p_1 not all principal minors are positive, hence we conclude again that this point does not solve the LMI).

Geometry of the LMI

The set of feasible solutions of considered LMI (the feasibility set) is denoted as follows

$$\mathcal{F} = \{x \in \mathbb{R}^M : F(x) = F_0 + \sum_{i=1}^{M} x_i F_i < 0\}$$

Due to the fact that LMI is defined in the space of its decision variables ($x \in \mathbb{R}^M$) it is possible to present the feasibility set as a geometrical shape in this space.

Some facts on matrix minors

- For the **positive (non-negative)** definiteness of $F(x)$ it is required that all of its diagonal minors to be positive (non-negative).
- For the **negative (non-positive)** definiteness of $F(x)$ it is required that its diagonal minors of odd minors to be negative (non-positive) and the minors of even degree to be positive (non-negative) respectively.
- It is straightforward to see that the diagonal minors are multi-variable polynomials of variables x_i. So the LMI set can be described as

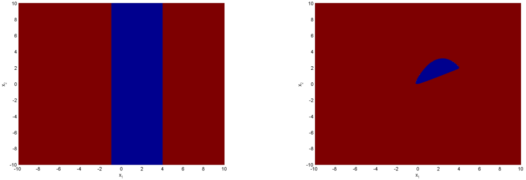
$$\mathcal{F}(x) = \{x \in \mathbb{R}^M : f_i(x) > 0, \ i = 1, .., M\}$$

which is a semi-algebraic set. Moreover, it is a convex set.

$$F(x_1, x_2) = \begin{bmatrix} x_1 - 4 & -x_2 + 2 & 0 \\ -x_2 + 2 & -1 & x_1 - x_2 \\ 0 & x_1 - x_2 & -x_1 - 1 \end{bmatrix} < 0$$

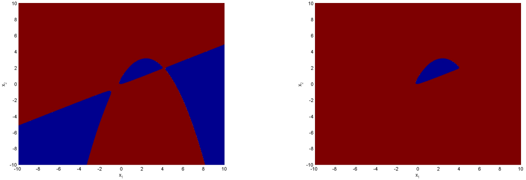
To find the feasibility region of the above LMI write the conditions for the diagonal minors of degree: first, second and third in variables $x_1,\ x_2$. So the minors become

$$\begin{cases} x_1 - 4 < 0 & \\ -1 < 0 & \text{first degree minors} \\ -x_1 - 1 < 0 & \text{- must be negative} \\ \hline -(x_1 - 4) - (-x_2 + 2)^2 > 0 & \\ -(-x_1 - 1) - (x_1 - x_2)^2 > 0 & \text{second degree minors} \\ (x_1 - 4)(-x_1 - 1) > 0 & \text{- must be positive} \\ \hline -(x_1 - 4)(-x_1 - 1) & \\ -(-x_2 + 2)^2(-x_1 - 1) & \text{third degree minor } (\det F(x)) \\ -(x_1 - 4)(x_1 - x_2)^2 < 0 & \text{- must be negative} \end{cases}$$



a) b)

Figure: The solutions for the first (a) and second (b) degree minors



a) b)

Figure: The solutions for the third (a) degree minors and the feasibility region for the considered LMI (b)

Conclusion

It is straightforward to see that the feasibility region is the $\bigcap$ of the regions which satisfy the constrains due to corresponding minors.

Matlab solution (script *test*0.*m* provides $x_1 = 1.6667$, $x_2 = 1.8333$). Refer to the figure

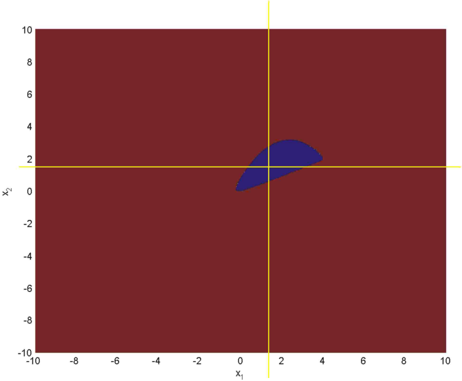


Figure: The feasibility region and the Matlab solution

Stabilization via state feedback

Consider the linear time-invariant system with one control input u_k in the form

$$x_{k+1} = Ax_k + Bu_k, \quad x_0 \neq 0, \quad x_k \in \mathbb{R}^n, \quad u_k \in \mathbb{R}^m$$

connected in feedback with the state feedback controller

$$u_k = Kx_k$$

This arrangement produces the following closed-loop system

$$x_{k+1} = (A + BK)x_k$$

Stabilization via state feedback

- The considered system can be tested for asymptotic stability using the Lyapunov method for any given controller K.
- as the result we have the following inequality

$$P > 0, \quad (A + BK)^T P(A + BK) - P < 0$$

or (for differential system)

$$P > 0, \quad (A + BK)^T P + P(A + BK) < 0$$

where P and K are variables (they are unknown).

Unfortunately, one can easily show that the resulting inequalities are not jointly convex on P and K.

Bilinear Matrix Inequality

Bilinear Matrix Inequality (BMI) has the following form

$$F(x,y) = F(x,y)^T = F_0 + \sum_{i=1}^{n} x_i F_i + \sum_{j=1}^{m} y_j G_j + \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j H_{ij} < 0$$

where

- $x = (x_1, \ldots, x_n)$, $x \in \mathbb{R}^n$ and $y = (x_1, \ldots, y_m)$, $y \in \mathbb{R}^m$ are the variables
- symmetric matrices F_0, F_i, $i = 1, \ldots, n$, G_j, $j = 1, \ldots, m$ and H_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, m$ are given data.

Remark

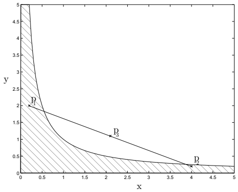
Unfortunately, BMIs are in general highly non-convex optimization problems, which can have multiple local solutions, hence solving a general BMI was shown to be $\mathcal{NP}$-hard problem

Bilinear Matrix Inequality

Consider the following bilinear inequality, i.e. non-convex set

$$1 - xy > 0 \tag{6}$$

To see this, consider two points on xy-plane which satisfy (6), e.g. $p_1 = (x_1, y_1) = (0.2, 2)$ and $p_2 = (x_2, y_2) = (4, 0.2)$



Obviously, the point in the half way between the two values, i.e.

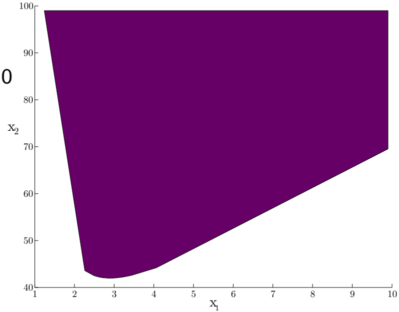
$$p_3 = \frac{1}{2}(0.2, 2) + \frac{1}{2}(4, 0.2) = (2.1, 1.1)$$

does not satisfy (6).

Can BMIs be written in the form of LMIs?

- Consider the following BMI

$$\begin{bmatrix} x_2(-13-5x_1+x_2) & x_2 & 0 \\ x_2 & x_1 & 0 \\ 0 & 0 & x_1(-13-5x_1+x_2)-x_2 \end{bmatrix} > 0$$



  - feasibility set is convex !
  - the LMI form can be obtained

$$\begin{bmatrix} -1+x_1 & 1 \\ 1 & -1-\frac{5}{18}x_1+\frac{1}{18}x_2 \end{bmatrix} > 0$$

An important fact from the matrix theory

If some matrix $F(x)$ is positive defined than $z^T F(x) z > 0$, $\forall z \neq 0$, $z \in \mathbb{R}^n$. Assume now that $z = My$ where M is any given nonsingular matrix, hence

$$z^T F(x) z > 0$$

implies that

$$y^T M^T F(x) M y > 0$$

This means that some rearrangements of the matrix elements do not change the feasible set of LMIs.

If the following LMI is feasible

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} < 0$$

then immediately the following LMI is feasible too

$$\begin{bmatrix} D & C \\ B & A \end{bmatrix} < 0$$

where

$$\begin{bmatrix} D & C \\ B & A \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

Change of variables

In the case of the Lyapunov inequalities, the fact that the nice inequalities previously obtained are functions of $X := P^{-1}$,and not P, suggests that we might start by rewriting the inequalities in terms of X.

$$X > 0, \quad X(A+BK)^T + (A+BK)X < 0$$

We then manipulate the second inequality by expanding the products

$$AX + XA^T + BKX + XK^TB^T < 0.$$

Change of variables

- introduce the new unknown $L = KX$
- to eliminate the matrix K, or, in other words, K can be explicitly expressed in terms of other unknowns, by solving the change of variable equation for the unknown K. This produces $K = LX^{-1}$
- finally, we have to solve the following inequality

$$X > 0, \quad AX + XA^T + BL + L^T B^T < 0,$$

Quadratic but convex inequality can be converted into the LMI form using the **Schur complement formula** given by the following Lemma.

Let $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times m}$ be symmetric matrices and $A > 0$ then

$$C + B^T A^{-1} B < 0$$

if and only if

$$U = \begin{bmatrix} -A & B \\ B^T & C \end{bmatrix} < 0 \quad \text{or, equivalently,} \quad U = \begin{bmatrix} C & B^T \\ B & -A \end{bmatrix} < 0$$

The matrix $C + B^T A^{-1} B$ is called the **Schur complement** of A in U. The identical result holds for a positive defined case.

Consider a controller design for discrete LRPs. It can be shown that the following LMI gives sufficient condition for stability along the pass

$$(\Phi + RK)^T W (\Phi + RK) - W < 0 \tag{7}$$

where $W > 0$ is block-diagonal matrix variable, Φ and R are given matrices identified in process state-space model as

$$\Phi = \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix}, \ R = \begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix}$$

and

$$K = \begin{bmatrix} K_1 & K_2 \\ K_1 & K_2 \end{bmatrix}$$

is the matrix to be found.

Applying the Schur complement formula to (7) yields

$$\begin{bmatrix} -W^{-1} & \Phi + RK \\ \Phi^T + K^T R^T & -W \end{bmatrix} < 0$$

The above form is still nonlinear due to the occurrence of terms W^{-1} and W (hence it can be stated in terms of BMI). To overcome this problem, introduce the substitution $P = W^{-1}$ and then multiply the result from the left and the right by $\mathrm{diag}(I, P)$ to obtain

$$\begin{bmatrix} -P & \Phi P + RN \\ P\Phi^T + N^T R^T & -P \end{bmatrix} < 0 \tag{8}$$

where $N = KP$. Now, it is straightforward to see that (8) is **numerically solvable**.

Elimination of a norm-bounded matrix

In robustness analysis, we often encounter the following terms

$$H\mathcal{F}E + E^T\mathcal{F}^TH^T \qquad (9)$$

where H, E are known real matrices of appropriate dimensions, and the matrix $\mathcal{F}$ represents parameter uncertainties which satisfies

$$\mathcal{F}^T\mathcal{F} \leqslant I \text{ or equivalently } \|\mathcal{F}\| \leqslant 1$$

Inequalities which consist of (9) can be transformed into the LMI with the following Lemma

Lemma

Let H, E be given real matrices of appropriate dimensions and $\mathcal{F}$ satisfy $\mathcal{F}^T\mathcal{F} \leqslant I$. Then for any $\epsilon > 0$ the following holds

$$H\mathcal{F}E + E^T\mathcal{F}^TH^T \leqslant \epsilon HH^T + \frac{1}{\epsilon}E^TE$$

Elimination of a norm-bounded matrix - proof

Since it is true that

$$\left(\epsilon^{\frac{1}{2}} H^T - \epsilon^{-\frac{1}{2}} \mathcal{F} E\right)^T \left(\epsilon^{\frac{1}{2}} H^T - \epsilon^{-\frac{1}{2}} \mathcal{F} E\right) \geqslant 0$$

then expansion of the above yields

$$\epsilon^{-1} E^T \mathcal{F}^T \mathcal{F} E + \epsilon H H^T \geqslant H \mathcal{F} E + E^T \mathcal{F}^T H^T$$

Next, observe that

$$\|\mathcal{F}\| \leqslant 1 \Leftrightarrow \lambda_{\max}(\mathcal{F}^T \mathcal{F}) \leqslant 1 \Leftrightarrow \mathcal{F}^T \mathcal{F} \leqslant I$$

hence

$$\epsilon H H^T + \frac{1}{\epsilon} E^T E \geqslant \epsilon^{-1} E^T \mathcal{F}^T \mathcal{F} E + \epsilon H H^T \geqslant H \mathcal{F} E + E^T \mathcal{F}^T H^T$$

and the proof is complete.

Elimination of variables

For certain specific matrix inequalities, if is often possible to eliminate some of the matrix variables.

Lemma

Let $\Psi \in \mathbb{R}^{q\times q}$ be a symmetric matrix and $P \in \mathbb{R}^{r\times q}$ and $Q \in \mathbb{R}^{s\times q}$ be real matrices then there exists a matrix $\Theta \in \mathbb{R}^{r\times s}$ such that

$$\Psi + P^T\Theta^T Q + Q^T\Theta P < 0$$

if and only if the inequalities

$$\mathcal{W}_P^T\Psi\mathcal{W}_P < 0 \text{ and } \mathcal{W}_Q^T\Psi\mathcal{W}_Q < 0$$

both hold, where $\mathcal{W}_P$ and $\mathcal{W}_Q$ are full rank matrices satisfying $\mathrm{Im}(\mathcal{W}_P) = \ker(P)$ and $\mathrm{Im}(\mathcal{W}_Q) = \ker(Q)$

It can also be used to eliminate variables from already formulated LMI. Since some variables can be eliminated, the computation burden can be reduced greatly.

Elimination of variables - an example

Consider again the stabilisation problem. The right-hand term can be rewritten as

$$\begin{bmatrix} -P & \Phi P + RN \\ P\Phi^T + N^T R^T & -P \end{bmatrix} = \begin{bmatrix} -P & \Phi P \\ P\Phi^T & -P \end{bmatrix} + \begin{bmatrix} R \\ 0 \end{bmatrix} N \begin{bmatrix} 0 & I \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} N^T \begin{bmatrix} R^T & 0 \end{bmatrix}$$

Using Elimination of Variables Lemma, we obtain

$$\mathcal{W}_R^T \begin{bmatrix} -P & \Phi P \\ P\Phi^T & -P \end{bmatrix} \mathcal{W}_R^T < 0, \quad \mathcal{W}_S^T \begin{bmatrix} -P & \Phi P \\ P\Phi^T & -P \end{bmatrix} \mathcal{W}_S^T < 0$$

where $\mathcal{W}_R = \mathrm{diag}(\ker(R), I)$ and $\mathcal{W}_S = \mathrm{diag}(I, 0)$. These two LMI conditions can be checked with less computation burden than the LMI condition provided in the Schur Complement example.

Illustrative computations have been performed for processes of prescribed order (n) and the results are listed in the below Table.

Table: Execution time comparison.

n	Previous example (CPU time)	This example (CPU time)
6	0.11	0.06
8	0.22	0.11
12	1.15	0.6
15	19.06	1.76
20	73.44	7.91

Note that all computations have been performed with LMI CONTROL TOOLBOX 1.0.8 under MATLAB 6.5. The MATLAB-files have been run on a PC with AMD Duron 600 MHz CPU and 128MB RAM.

Control problems solved with LMIs - details

- The discrete system state-space equation

$$x_{k+1} = Ax_k + Bu_k$$

- Lyapunov inequality for discrete system

$$A^T PA - P < 0, \; P > 0$$

- Controller design:
  The closed loop for the discrete case $u_k = Kx_k$ (K is the controller to be designed)
- The stabilization condition for the discrete case

$$(A + BK)^T P(A + BK) - P < 0, \; P > 0$$

not the LMI condition since the matrix variables are multiplied

It requires some operations

1. The stabilization condition for the discrete case $(A+BK)^T P(A+BK) - P < 0,\ P > 0$ - **not the LMI condition since the variable matrices are multiplied**
2. Schur complement to get
$$\begin{bmatrix} -P & A^T + K^T B^T \\ A + BK & -P^{-1} \end{bmatrix} < 0,\ P > 0$$
3. Left- and right- multiplication by $\mathrm{diag}(P^{-1}, I)$ and set $Q = P^{-1}$
$$\begin{bmatrix} -Q & QA^T + QK^T B^T \\ AQ + BKQ & -Q \end{bmatrix} < 0,\ Q > 0$$
4. Setting $K = NQ^{-1}$ to obtain finally the LMI
$$\begin{bmatrix} -Q & QA^T + N^T B^T \\ AQ + BN & -Q \end{bmatrix} < 0,\ Q > 0$$

Operations to be performed for CTLTI systems

1. the Closed loop Lyapunov inequality

$$(A + BK)^T P + P(A + BK) < 0,\ P > 0$$

2. the congruence transformation (left- and right- multiplication) by P^{-1} to get

$$P^{-1}(A + BK)^T + (A + BK)P^{-1} < 0,\ P^{-1} > 0$$

3. set $Q = P^{-1}$

$$QA + QK^T B^T + AQ + BKQ < 0,\ Q > 0$$

4. finally set $K = NQ^{-1}$ to obtain the following LMI

$$QA + N^T B^T + AQ + BN < 0,\ Q > 0$$

Standard LMI problems

The LMI software can solve the LMI problems formulated in three different forms:

- feasibility problem,
- linear optimization problem,
- generalized eigenvalue minimization problem.

Feasibility problem

A feasibility problem is defined as follows

Definition

Find a solution $x = (x_1, \ldots, x_n)$ such that

$$F(x) > 0 \tag{10}$$

or determine that the LMI (10) is infeasible.

A typical situation for the feasibility problem is a stability problem where one has to decide if a system is stable or not (an LMI is feasible or not).

Linear objective minimization problem

Definition

Minimize a linear function $c^T x$ ($x = (x_1, \ldots, x_n)$), where $c \in \mathbb{R}^n$ is a given vector, subject to an LMI constraint (10) or determine that the constraint is infeasible. Thus the problem can be written as

$$\begin{aligned} \min \ & c^T x \\ \text{subject to} \ & F(x) > 0 \end{aligned}$$

This problem can appear in the equivalent form of minimizing the maximum eigenvalue of a matrix that depends affinely on the variable x, subject to an LMI constraint (this is often called EVP)

$$\begin{aligned} \min \ & \lambda \\ \text{subject to} \ & \lambda I - F(x) > 0 \end{aligned}$$

Generalized eigenvalue problem

The generalized eigenvalue problem (GEVP) allows us to minimize the maximum generalized eigenvalue of a pair of matrices that depend affinely on the variable $x = (x_1, \ldots, x_n)$. The general form of GEVP is stated as follows

$$\min \quad \lambda$$

$$\text{subject to} \quad \begin{cases} A(x) < \lambda B(x) \\ B(x) > 0 \\ C(x) < D(x) \end{cases} \tag{11}$$

where $C(x) < D(x)$ and $A(x) < \lambda B(x)$ denote set of LMIs. It is necessary to distinguish between the standard LMI constraint, i.e.

$$C(x) < D(x)$$

and the LMI involving λ (called the linear-fractional LMI constraint)

$$A(x) < \lambda B(x)$$

which is quasi-convex with respect to the parameters x and λ. However, this problem can be solved by similar techniques as those for previous problems.

The stability margins

Known facts from matrix theory

For any symmetric matrix Q

- $\lambda_{\text{min}}(Q)I \leqslant Q \leqslant \lambda_{\text{max}}(Q)I$
- $\lambda_{\text{max}}(Q + I) \leqslant \lambda_{\text{max}}(Q) + \lambda_{\text{max}}(I)$

Degree of stability

The LTI system has a degree of stability equal to $q > 0$, iff $\exists P > 0$ such that

$$[A + qI]^T P[A + qI] - P < 0$$

for DTLTI or

$$[A + qI]^T P + P[A + qI] < 0$$

for CTLTI

The stability margins

For CTLTI systems we have

$$\begin{bmatrix} -Q & (1+q)(QA^T + N^T B^T) \\ (1+q)(AQ + BN) & -Q \end{bmatrix} < 0, \; Q > 0$$

$K = NQ^{-1}$

Remark

Since$(1+q)$ is a scalar, so for any matrix, say T, holds $(1+q)T = T(1+q)$

To get the GEVP condition, the follow the steps (on the next slide)

The stability margins - GEVP

- $$\begin{bmatrix} -Q & QA^T + N^T B^T \\ AQ + BN & -Q \end{bmatrix} < -q \begin{bmatrix} 0 & QA^T + N^T B^T \\ AQ + BN & 0 \end{bmatrix}, \; Q > 0$$

- multiply it by q^{-1} (positive since $q > 0$) and set $\lambda = q^{-1}$ and then by -1 to obtain

$$\begin{bmatrix} 0 & QA^T + N^T B^T \\ AQ + BN & 0 \end{bmatrix} < \lambda \begin{bmatrix} Q & -QA^T - N^T B^T \\ -AQ - BN & Q \end{bmatrix}, \; Q > 0$$

- write it as the GEVP

$$\min \lambda \text{ s.t.}$$

$$\begin{cases} \begin{bmatrix} 0 & QA^T + N^T B^T \\ AQ + BN & 0 \end{bmatrix} < \lambda \begin{bmatrix} Q & -QA^T - N^T B^T \\ -AQ - BN & Q \end{bmatrix} \\ \begin{bmatrix} -Q & QA^T + N^T B^T \\ AQ + BN & -Q \end{bmatrix} < 0 \end{cases}$$

- $K = NQ^{-1}$, $q = \lambda^{-1}$

- note that the additional constraint is just the stability condition in this case

$\mathcal{D}$-stability (Poles placement)

References

- LMI MATLAB Control toolbox manual
- M. Chilali and P. Gahinet, "H_∞ design with pole placement constraints: an LMI approach," IEEE Transactions on Automatic Control, vol. 41, no. 3, pp. 358–367, 1996.
- M. Chilali, P. Gahinet, and P. Apkarian, "Robust pole placement in LMI regions," IEEE Transactions on Automatic Control, vol. 44, no. 12, pp. 2257–2270, 1999.
- D. Henrion, M. Sebek, and V. Kucera, "Robust pole placement for second-order systems: an lmi approach," Proceedings of the IFAC Symposium on Robust Control Design, 2003. LAAS-CNRS Research Report No. 02324, July 2002.

LMI regions

LMI region is any subset $\mathcal{D}$ of the complex plane that can be defined as

$$\mathcal{D} = \{z \in C : L + zM + \bar{z}M^T < 0\} \qquad (\%)$$

where L and M are real matrices and $L = L^T$
The matrix-valued function

$$f_{\mathcal{D}}(z) = L + zM + \bar{z}M^T$$

is called the characteristic function of $\mathcal{D}$

LMI regions

A real matrix A is $\mathcal{D}$-stable, i.e. has all eigenvalues inside the $\mathcal{D}$ region iff there exists a symmetric matrix $X > 0$ such that the following LMI holds

$$L \otimes X + M \otimes (XA) + M^T \otimes (A^T M) < 0 \qquad (\$)$$

where $\otimes$ denotes the Kronecker product

Very important result !!! - indeed, this generalizes all we said about the stability

Common known $\mathcal{D}$ regions

- half-plane $Re(z) < -\alpha$: $z + \bar{z} + 2\alpha < 0$
- special case of the above i.e. $Re(z) < 0$: $z + \bar{z} < 0$ - the stability region for the differential system described by A
- disc centered at $(-q, 0)$ with radius r

$$\begin{bmatrix} -r & q+z \\ q+\bar{z} & -r \end{bmatrix} < 0$$

- ellipse centered at $(-q, 0)$ with radiuses a-horizontal and b-vertical

$$\begin{bmatrix} -2a & -2g+(1+a/b)z+(1-a/b)\bar{z} \\ -2g+(1-a/b)z+(1+a/b)\bar{z} & -2a \end{bmatrix}$$
$$= \begin{bmatrix} -2a & -2g \\ -2g & -2a \end{bmatrix} + z \begin{bmatrix} 0 & (1+a/b) \\ (1-a/b) & 0 \end{bmatrix} + \bar{z} \begin{bmatrix} 0 & (1-a/b) \\ (1+a/b) & 0 \end{bmatrix}$$
$$= L + zM + \bar{z}M^T < 0$$

- conic sector with apex at the origin and inner angle (see reference)
- any intersection(s) of the above

Application in control

Note, that it is now easy to find the LMI condition which checks, if matrix eigenvalues lay inside chosen region - **weak?** Yes, but ...
Instead of A write it in the closed loop configuration i.e. $A + BK$ - we get the way to drive A to have eigenvalues inside chosen region using controller K- **strong enough!**
So, the procedure is as follows

- choose $\mathcal{D}$
- write it as LMI region (%)
- write ($)
- use some linear algebra operations to get programmable LMI
- solve it using your favorite software

$\mathcal{D}$-stabilization - example

1. choose ellipse

$$\begin{bmatrix} -2a & -2g \\ -2g & -2a \end{bmatrix} + z \begin{bmatrix} 0 & (1+a/b) \\ (1-a/b) & 0 \end{bmatrix} + \bar{z} \begin{bmatrix} 0 & (1-a/b) \\ (1+a/b) & 0 \end{bmatrix}$$
$$= L + zM + \bar{z}M^T < 0$$

2. set $\mathcal{A} = A + BK$
3. condition for the closed loop system

$$L \otimes X + M \otimes (X\mathcal{A}) + M^T \otimes (\mathcal{A}^T X) < 0$$

which can be rewritten as

$$\begin{bmatrix} -2aX & (*) \\ -2gX + (1+\frac{a}{b})\mathcal{A}^T X + (1-\frac{a}{b})X\mathcal{A} & -2aX \end{bmatrix} < 0$$

4. set $\mathcal{A} = A + BK$ to obtain

$$\begin{bmatrix} -2aX & (*) \\ -2gX + (1+\frac{a}{b})(A+BK)^T X + (1-\frac{a}{b})X(A+BK) & -2aX \end{bmatrix} < 0$$

not LMI - again X and K multiplied

5. Pre and post multiply it by $diag(X^{-1}, X^{-1})$ and set $Y = X^{-1}$ to obtain

$$\begin{bmatrix} -2aY & (*) \\ -2gY+(1+\frac{a}{b})YA^T+YK^TB^T+(1-\frac{a}{b})AY+BKY & -2aY \end{bmatrix} < 0$$

which still isn't the LMI

6. set $K = NY^{-1}$ to obtain the LMI

$$\begin{bmatrix} -2aY & (*) \\ -2gY+(1+\frac{a}{b})YA^T+N^TB^T+(1-\frac{a}{b})AY+BN & -2aY \end{bmatrix} < 0$$

Robust stability

DTLTI Uncertain system

$$x_{k+1} = Ax_k, \quad x_0 = 0, \quad x_k \in \mathbb{R}^n, \quad A \in \mathcal{A}$$

where $\mathcal{A}$ is an arbitrary **closed convex set**

Robust Stability

The DTLTI uncertain system is said to be robustly stable if it is asymptotically stable for all $A \in \mathcal{A}$.

Problem

The set of all (discrete-time) stable matrices is not a convex set.

Problem with robust stability for discrete system

Let us consider a set formed from 2 vertices and assume that they are

$$A_1 = \begin{bmatrix} 0.5 & 2 \\ 0 & 0.5 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.5 & 0 \\ 2 & 0.5 \end{bmatrix}$$

Based on well known fact that stability in the discrete case is guaranteed if and only if all eigenvalues of a system matrix lie in the interior of the unit circle, it can be seen that the matrices A_1 and A_2 are stable ($\lambda_{\max}(A_1) = 0.5$ and $\lambda_{\max}(A_2) = 0.5$). However, a convex combination yields

$$A = 0.5A_1 + 0.5A_2 = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}$$

and $\lambda_{\max}(A) = 1.5$. This means that A is unstable.

Uncertainty

Main models of uncertainty

- **norm-bounded**
- **polytopic**
- **affine**

Norm-bounded model of uncertainty

This model of uncertainty corresponds to a system which matrices uncertainty are modelled as **an additive perturbation** to the nominal system matrices. Therefore a system is said to be subjected to norm-bounded parameter uncertainty if matrices of such a system can be written in the form

$$M = M_0 + \Delta M = M_0 + H\mathcal{F}E$$

where H and E are some known constant matrices with compatible dimensions and M_0 defines the nominal system. $\mathcal{F}$ is an unknown, constant matrix which satisfies

$$\mathcal{F}^T\mathcal{F} \leqslant I$$

Norm-bounded model of uncertainty

Important: The inequality

$$\mathcal{F}^T\mathcal{F} \leqslant I$$

represents a convex set.
To see this, apply the Schur complement formula to obtain

$$\mathcal{F}^T\mathcal{F} \leqslant I \quad \Leftrightarrow \quad \begin{bmatrix} I & \mathcal{F} \\ \mathcal{F}^T & I \end{bmatrix} \geqslant 0$$

Polytopic model of uncertainty

This model of uncertainty corresponds to a system which matrices range in the **polytope of matrices**. This means that each system matrix M is only known to lie in a given fix polytope of matrices described by

$$M \in \mathrm{Co}\left(M_1, M_2, \ldots, M_h\right)$$

where Co denotes the convex hull. Then, for positive $i = 1, 2, \ldots, h$, M can be written as

$$M := \left\{ X : X = \sum_{i=1}^{h} \alpha_i M_i, \quad \alpha_i \geqslant 0, \quad \sum_{i=1}^{h} \alpha_i = 1 \right\}$$

Polytopic model of uncertainty

As a simple example, the polytope formed from 4 vertices: M_1, M_2, M_3 and M_4 is depicted below

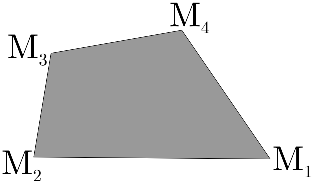


Figure: A polytope

Affine model of uncertainty

This model of uncertainty corresponds to a system which matrices are modelled as a collection of fixed affine functions of some varying parameters $p_1, \ldots, p_k$ i.e. each matrix can be written in the form

$$M(p) = M_0 + p_1 M_1 + \ldots + p_k M_k \tag{12}$$

where $M_i \ \forall i = 0, 1, \ldots, k$ are given. Parameter uncertainty is described with range of parameter values. It means that each parameter p_i ranges between two known extremal values $\underline{p_i}$ (minimum) and $\overline{p_i}$ (maximum), therefore it can be written as

$$\underline{p_i} \leqslant p_i \leqslant \overline{p_i}$$

Furthermore, the set of uncertain parameters is

$$\Delta \triangleq \left\{p = (p_1, p_2, \ldots, p_k) : \underline{p_i} \leqslant p_i \leqslant \overline{p_i},\ i = 1, \ldots, k\right\}$$

and the set of corners of uncertainty region Δ_0 is defined as

$$\Delta_0 \triangleq \left\{p = (p_1, p_2, \ldots, p_k) : p \in \{\underline{p_i}, \overline{p_i}\},\ i = 1, \ldots, k\right\}$$

Affine model of uncertainty

As an example of a set of uncertain parameters, consider 3 parameters: p_1, p_2, p_3 whose values range in the parameter box formed by their extremal values in 3-D parameter space

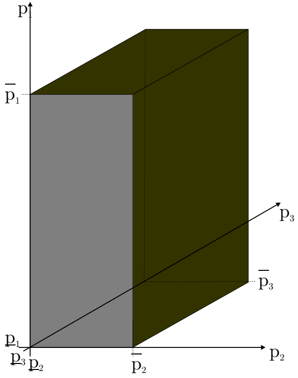


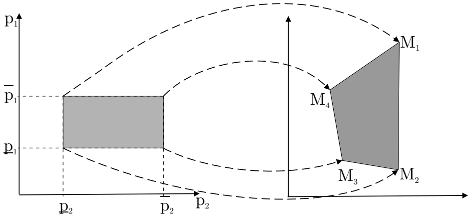
Figure: 3-D parameter space

Form affine to the polytopic form

It is clear that $M(p)$ is an affine function in $p = (p_1, p_2, \ldots, p_k)$, thus it maps these corners to the polytope of vertices. In this case each vertex can be determined $\forall p \in \Delta_0$ with the formula below

$$M_i = M_0 + p_1 M_1 + \ldots + p_k M_k$$

where $i = 1, .., 2^k$.



Norm-bounded uncertainty

- The system plant $\Phi := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$
- Uncertainties
  $\Delta\Phi := \begin{bmatrix} \Delta A & \Delta B \\ \Delta C & \Delta D \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(k) \begin{bmatrix} E_1 & E_2 \end{bmatrix}$
- Unknown matrix $||F(k)|| < 1$
- The uncertain system plant matrix is $\Phi + \Delta\Phi$

Note on an unknown matrix

In general

$$||F(k)|| \leqslant \gamma \Leftrightarrow \sigma_{\max}^2(F(k)) = \lambda_{\max}(F(k)^T F(k)) \leqslant \gamma^2 I$$
$$\Leftrightarrow F(k)^T F(k) \leqslant \gamma^2 I$$

Norm-bounded uncertainty

Discrete case - controller design (so we want to get the condition for computing K)

- For simplicity take $F(k) = F$ - no loose of generalization
- The Lyapunov inequality for the closed loop uncertain system

$$(A + H_1 F E_1 + BK + H_1 F E_2 K)^T P (A + H_1 F E_1 + BK + H_1 F E_2 K) - P < 0, \; P > 0$$

- Schur complement gives

$$\begin{bmatrix} -P & A^T + E_1^T F^T H_1^T + K^T B^T + K^T E_2^T F^T H_1^T \\ A + H_1 F E_1 + BK + H_1 F E_2 K & -P^{-1} \end{bmatrix} < 0$$

- and then

$$\begin{bmatrix} -P & A^T + K^T B^T \\ A + BK & -P^{-1} \end{bmatrix} + \begin{bmatrix} 0 & E_1^T F^T H_1^T + K^T E_2^T F^T H_1^T \\ H_1 F E_1 + H_1 F E_2 K & 0 \end{bmatrix} < 0, \; P > 0$$

Towards LMI formulation

- Furthermore

$$\begin{bmatrix} -P & A^T+K^TB^T \\ A+BK & -P^{-1} \end{bmatrix} + \begin{bmatrix} E_1^T+K^TE_2^T \\ 0 \end{bmatrix} \begin{bmatrix} F^T \end{bmatrix} \begin{bmatrix} 0 & H_1^T \end{bmatrix} + \begin{bmatrix} 0 \\ H_1 \end{bmatrix} \begin{bmatrix} F \end{bmatrix} \begin{bmatrix} E_1+E_2K & 0 \end{bmatrix} < 0,\ P > 0$$

- Apply the appropriate Lemma to obtain

$$\begin{bmatrix} -P & A^T+K^TB^T \\ A+BK & -P^{-1} \end{bmatrix} + \epsilon^{-1} \begin{bmatrix} E_1^T+K^TE_2^T \\ 0 \end{bmatrix} \begin{bmatrix} E_1+E_2K & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ H_1 \end{bmatrix} \begin{bmatrix} 0 & H_1^T \end{bmatrix} < 0,\ P > 0$$

- write it as (since $\epsilon > 0$ is a scalar)

$$\begin{bmatrix} -P & A^T+K^TB^T \\ A+BK & -P^{-1} \end{bmatrix} + \begin{bmatrix} E_1^T+K^TE_2^T \\ \epsilon H_1 \end{bmatrix} \epsilon^{-1} I \begin{bmatrix} E_1+E_2K & \epsilon H_1^T \end{bmatrix} < 0, \; P > 0$$

- Apply the Schur complement again

$$\left[\begin{array}{cc|c} -P & A^T+K^TB^T & E_1^T+K^TE_2^T \\ A+BK & -P^{-1} & \epsilon H_1 \\ \hline E_1+E_2K & \epsilon H_1^T & -\epsilon I \end{array}\right] < 0, \; P > 0$$

still not the LMI

- congruence by $diag(P^{-1}, I, I)$ and set $Q = P^{-1}$

$$\begin{bmatrix} -Q & QA^T + QK^TB^T & QE_1^T + QK^TE_2^T \\ AQ + BKQ & -Q & \epsilon H_1 \\ E_1Q + E_2KQ & \epsilon H_1^T & -\epsilon I \end{bmatrix} < 0, \; Q > 0$$

- finally set $K = NQ^{-1}$ to obtain the LMI

$$\begin{bmatrix} -Q & QA^T + N^TB^T & QE_1^T + N^TE_2^T \\ AQ + BN & -Q & \epsilon H_1 \\ E_1Q + E_2N & \epsilon H_1^T & -\epsilon I \end{bmatrix} < 0, \; Q > 0$$

Thank you very much for your attention