Note on controller design with LMIs

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Outline

Controller design problem

Consider DTLTI system ($D = 0$)

$$\begin{aligned} x(k+1) =& Ax(k) + B_w w(k) + Bu(k), \\ z(k) =& C_z x(k) + D_{zw} w(k) + D_{zu} u(k), \\ y(k) =& Cx(k) + D_{yw} w(k) \end{aligned}$$

where w is exogenous input (disturbance), y is measured output, z is signal related to the performance.
3 types of control

- when no restriction are imposed on C and D_{yw} then we have
  - static output control problem ($u(k) = Ky(k)$)
  - dynamic output control problem when K is dynamic system, i.e. $K(k)$
- when $C = I$, $D_{yw} = 0$ and $u(t) = Kx(k)$ then we deal with **state feedback control**.

State feedback controller

Since $C = I$, $D_{yw} = 0$ and $u(tk) = Kx(k)$ the closed loop matrices are

$$A_{cl} = A + BK,\ B_{cl} = B_w,\ C_{cl} = C_z + D_{zu}K,\ D_{cl} = D_{zw}$$

Stability of the closed loop system

The closed–loop system is stable iff there exists $P > 0$ such that

$$A_{cl}^T P A_{cl} - P < 0$$

or $X > 0$ such that

$$\begin{bmatrix} -X & XA^T + L^T B^T \\ AX + BL & -X \end{bmatrix} < 0$$

where $K = LX^{-1}$ and $X = P^{-1}$.

Output feedback controller

Since there is no restrictions on C and $u(t) = Ky(t)$ then

$$A_{cl} = A + BKC, \; B_{cl} = B_w, \; C_{cl} = C_z + D_{zu}KC, \; D_{cl} = D_{zw}$$

where $D_{yw} = 0$ for simplicity only.

Our problem is to solve the following MI (rewrite it in terms of LMIs)

$$(A + BKC)^T P(A + BKC) - P < 0$$

or after Schur complement formula

$$\begin{bmatrix} -P & (A + BKC)^T P \\ P(A + BKC) & -P \end{bmatrix} < 0$$

Static output feedback controller

Let $X = P^{-1}$ and pre- and post-multiply the last inequality by $diag(X, X)$ we get

$$\begin{bmatrix} -X & (AX + BKCX)^T \\ AX + BKCX & -X \end{bmatrix} < 0$$

The term $BKCX$is nonlinear since it contains the product of variables K and X. To overcome this we introduce the new condition: $CX = NC$. However this can be very restrictive.

Using $CX = NC$ and letting $Y = KN$, we get the result.

Static output feedback controller

Theorem

There exists a static output controller if there exist $X > 0$ and N and a matrix Y such that the following hold:

$$\begin{bmatrix} -X & (AX + BYC)^T \\ AX + BYC & -X \end{bmatrix} < 0, \quad CX = NC$$

The controller gain is given by $K = YN^{-1}$

The first paper on this transformation: Crusius, C. A. R., & Trofino, A. (1999). *Sufficient LMI conditions for output feedback control problems*. IEEE Transactions on Automatic Control, 44 (5), 1053–1057.

Static output feedback controller

The following lemma will be used in developing our results.

Finsler's lemma

Let $\zeta \in \mathbb{R}^n$, $P \in \mathbb{R}^{n \times n}$ a symmetric and positive-definite matrix and a matrix $S \in \mathbb{R}^{m \times n}$ such that $rank(S) = r < n$, then the following statements are equivalent:

- $\zeta^T P \zeta < 0$, $\forall \zeta \neq 0$ and $S\zeta = 0$;
- $\exists X \in \mathbb{R}^{n \times m}$ such that

$$P + XS + S^T X^T < 0$$

Also, let us now assume that the output matrix C is full row rank. This means that there exists a transformation, T_l (not unique) such that the following holds:

$$CT_l = [I \;\; 0]$$

Static output feedback controller

Assume that the matrix C is full rank. Then T_l can be computed as follows

$$T_l = [C^T(CC^T)^{-1} \;\; C^\perp]$$

where $C^\perp$ is the orthogonal basis of the null space of the matrix C (Matlab: `Tl=[C'*inv(C*C') null(C)]`).

If there exists $P > 0$, G and L with the following structures

$$G = \begin{bmatrix} G_1 & 0 \\ G_2 & G_3 \end{bmatrix}, L = [L_1 \; 0]$$

such that the following LMI holds

$$\begin{bmatrix} P - G - G^T & (AT_lG + BL)^T T_l^{-T} \\ T_l^{-1}(AT_lG + BL) & -P \end{bmatrix} < 0$$

then the controlled system is stable and $K = L_1 G_1^{-1}$.

Static output feedback controller

To prove it, observe that the matrix C is full row rank, which implies that there exists a matrix T_I, and the structure of the matrix L, we get:

$$L = [L_1 \; 0]$$

Using the expression of the controller, i.e. $K = L_1 G_1^{-1}$ we obtain

$$L = [KG_1 \; 0]$$

and therefore

$$L = K[I \; 0] \begin{bmatrix} G_1 & 0 \\ G_2 & G_3 \end{bmatrix} = KCT_I G$$

Static output feedback controller

If $R = T_I P T_I$ then $P = T_I^{-1} R T_I^{-T}$ and therefore

$$\begin{bmatrix} T_I^{-1} R T_I^{-T} - G - G^T & (AT_I G + BL)^T T_I^{-T} \\ T_I^{-1}(AT_I G + BL) & -T_I^{-1} R T_I^{-T} \end{bmatrix} < 0$$

Next pre- and post-multiply the above by $diag(T_I, T_I)$ to obtain

$$\begin{bmatrix} P - T_I(G + G^T)T_I & T_I^T G^T T_I (A + BKC)^T \\ (A + BKC) T_I G T_I^T & -P \end{bmatrix} < 0$$

Define M,X and H as follows

$$M = \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix}, X = \begin{bmatrix} T_I G T_I^T \\ 0 \end{bmatrix}, H = [-I \;\; (A + BKC)^T]$$

and the last inequality can be written as

$$M + XH + H^T X^T < 0$$

Furthermore, we know that

$$x(k+1) = (A + BKC)x(k)$$

and hence the closed-loop dynamics can be written as

$$H\zeta = 0,\ \zeta = [x^T(k+1)\ \ x^T(k)]^T$$

Application of Finsler's lemma

Finsler's lemma shows

$$M + XH + H^T X^T < 0 \ \Leftrightarrow\ \zeta^T M \zeta < 0$$

where $\zeta^T M \zeta < 0$ is

$$[x^T(k+1)\ \ x^T(k)] \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} x(k+1) \\ x(k) \end{bmatrix} < 0$$

and stability is guaranteed.

Output controller (full order)

Consider CTLTI system

$$\begin{aligned}\dot{x}(t) =& Ax(t) + B_w w(t) + Bu(t),\\ z(t) =& C_z x(t) + D_{zw} w(t) + D_{zu} u(t),\\ y(t) =& Cx(t) + D_{yw} w(t)\end{aligned}$$

where w is exogenous input (disturbance), y is measured output, z is signal related to the performance.

A dynamic controller (K) is of the form

$$\begin{aligned}\dot{x}_c(t) =& A_c x_c(t) + B_c y(t),\\ u(t) =& C_c x_c(t) + D_c y(t),\end{aligned}$$

Output controller, cont'd

The controlled system admits the realization

$$\begin{aligned}\dot{x}_{cl}(t) =& A_{cl}x_{cl}(t) + B_{cl}w(t),\\ z(t) =& C_{cl}x_{cl}(t) + D_{cl}w(t),\end{aligned}$$

where

$$\begin{aligned}A_{cl} =& \begin{bmatrix} A+BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}, B_{cl} = \begin{bmatrix} B_w+BD_cC \\ B_cC \end{bmatrix},\\ C_{cl} =& [C_z+D_{zu}D_cC \;\; D_{zu}C_c],\; D_{cl} = D_{zw}+D_{zu}D_cD_{yw}\end{aligned}$$

Output controller, cont'd

Obviously, we have

$$\begin{bmatrix} A+BDcC & BC_c \\ B_cC & A_c \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$$

which has the same forme as for static output controller.

Problem

Is there exist transformation that allows us to reformulate the stability problem as LMI?

Output controller, cont'd

Suppose that the Lyapunov matrix P and its inverse are partitioned into blocks as

$$P = \begin{bmatrix} Y & N^T \\ N & ? \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} X & M \\ M^T & ? \end{bmatrix},$$

where '?' is used to denote block entries in these matrices that are not involved in the derivations that follow. Also introduce

$$\Pi_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, \; \Pi_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix},$$

where $P\Pi_1 = \Pi_2$.

Change of variables

Let us now define the change of controller variables as follows

$$\begin{aligned}\mathcal{A} =& NA_c M^T + NB_c CX + YBC_c M^T + Y(A + BD_c C)X,\\ \mathcal{B} =& NB_c + YBD_c\\ \mathcal{C} =& C_c M^T + D_c CX,\\ \mathcal{D} =& D_c\end{aligned}$$

Details of this transformation: C.W. Scherer, P.Gahinet, and M. Chilali:*Multiobjective output-feedback control via LMI optimization*, IEEE Trans. Automatic Control, vol.42, no. 7, pp. 896–911, 1997.

The motivation for this transformation lies in the following identities:

$$\Pi_1^T P A_{cl} \Pi_1 = \Pi_2^T A_{cl} \Pi_1 = \begin{bmatrix} AX + B\mathcal{C} & A + B\mathcal{D}C \\ \mathcal{A} & YA + \mathcal{B}C \end{bmatrix},$$

$$\Pi_1^T P B_{cl} \Pi_1 = \Pi_2^T B_{cl} \Pi_1 = \begin{bmatrix} B_w + B\mathcal{D}F \\ YB + \mathcal{B}F \end{bmatrix}$$

$$C_{cl} \Pi_1 = [CX + D_{zu}\mathcal{C} \quad C_z + D_{zu}\mathcal{D}C]$$

$$\Pi_1^T P \Pi_1 = \Pi_1^T \Pi_1 = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}$$

We are mainly interested in solving $A_{cl}^T P + P A_{cl} < 0$. However, other objectives (control performance) are important too.

A systematic procedure for obtaining the corresponding controller matrices

1. Compute the singular value decomposition (SVD) of $I - XY$ to obtain the matrices U_1, V_1 such that $I - XY = U_1 \Sigma_1 V_1^T$.
2. Choose the matrices M and N as $M = U_1 \Sigma_1^{\frac{1}{2}}$, $N = \Sigma_1^{\frac{1}{2}} V_1^T$.
3. Compute the matrices of the controller using

$$
\begin{aligned}
D_c =& \mathcal{D} \\
C_c =& (\mathcal{C} + D_c C X) M^{-T}, \\
B_c =& N^{-1} (\mathcal{B} - Y B D_c), \\
A_c =& N^{-1} (\mathcal{A} - N B_c C X - Y B C_c M^T \\
& - Y (A + B D_c C) X) M^{-T}.
\end{aligned}
$$

Thank you very much for your attention