Example Exam Problems in Operations Research 2

The exam will consist of 4-5 problems of a nature and difficulty level similar to the example problems below.

1. Given is an integer linear programming problem:

$$\begin{array}{ll} \text{maximize} & 3x_1 + 4x_2 \\ \text{subject to} & x = (x_1, x_2) \in \mathbb{R}^2 \\ \text{with constraints} & 2x_1 + 3x_2 \leq 6 \\ & 3x_1 + 2x_2 \leq 6 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

   (a) Solve this problem graphically.

   (b) Using the simplex method, determine the optimal solution of this problem, ignoring the integer constraints of the variables.

   (c) For the solution determined in point (a), perform a single iteration of the Gomory cutting plane method.

   (d) Is the solution obtained after this single iteration the optimal solution of the original problem? Justify your answer.

   (e) Plot the solutions determined in points (a) and (b) on a diagram.

2. Given is a problem of maximizing the function f on a finite set G. Additionally, a certain family $\mathbf{G}$ of subsets of the set G is given, containing all single-element sets, and an upper-bound function φ is defined on $\mathbf{G}$. Which of the following statements are true:

   (a) On all single-element sets, the function φ takes the same value.

   (b) $x \in D \in \mathbf{G} \Longrightarrow f(x) < \varphi(D)$.

   (c) If during the implementation of the branch-and-bound method a set $D \in \mathbf{G}$ is divided into two disjoint subsets $D_1, D_2 \in \mathbf{G}$ and it is determined that $\varphi(D_1) \leq \varphi(D_2)$, then the set D_1 is no longer considered.

   (d) If during the implementation of the branch-and-bound method it is determined that $f(x) = \varphi(D)$ for some $x \in D \in \mathbf{G}$, then x is a candidate for the optimal solution.

3. Provide an example of an upper-bound function for an integer linear programming problem:

$$\begin{array}{ll} \text{maximize} & c^\top x \\ \text{subject to} & Ax \leq b \\ & 0 \leq x \leq h \\ & x \in \mathbf{Z}^n \end{array}$$

4. Given is an integer linear programming problem:

$$\begin{array}{ll} \text{maximize} & c^\top x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \\ & x \in \mathbf{Z}^n \end{array} \qquad \text{(ILP)}$$

   and the associated problem (LP) obtained from (ILP) by removing the integer constraints on the variables. Which of the following statements are true:

(a) Every feasible solution of problem (ILP) is a feasible solution of problem (LP).

(b) Every optimal solution of problem (ILP) is a feasible solution of problem (LP).

(c) If there exists a feasible solution to problem (LP), then there exists a feasible solution to problem (ILP).

(d) If there exists an optimal solution to problem (LP), then there exists an optimal solution to problem (ILP).

(e) If there is exactly one optimal solution to problem (LP), then there is exactly one optimal solution to problem (ILP).

(f) The optimal value of the objective function for problem (ILP) does not exceed the optimal value of the objective function for problem (LP).

All answers must be justified.

5. The transportation problem is defined by the following transportation table, which specifies the warehouse supplies and store demands, as well as the unit transportation costs:

	B_1	B_2	B_3	B_4	
A_1	3	6	7	5	14
A_2	8	4	1	2	10
A_3	7	4	2	3	6
	8	7	6	9	

(a) Using the northwest corner method, determine the basic feasible solution.

(b) For this solution, present the system of equations that the dual variables must satisfy.

(c) Determine the dual variables from this system.

(d) Calculate the negative reduced costs.

(e) Is this solution optimal? If not, perform a basis exchange according to the transportation algorithm.

6. Explain why the transportation algorithm for a transportation problem with integer supplies and demands yields an optimal integer solution.

7. Perform one iteration of the Hungarian method for the assignment problem with the following cost matrix:

	S_1	S_2	S_3	S_4
P_1	11	7	10	6
P_2	9	12	7	6
P_3	13	14	9	8
P_4	15	8	10	7

8. An undirected network (V, E, c) is given.

(a) Provide the definition of the minimum spanning tree.

(b) What conditions must this network satisfy for such a tree to exist?

9. A complete undirected graph $G = (V, E)$ is given, whose vertices are placed in the plane $\mathbf{R}^2$ and have coordinates $(0,0)$, $(0,3)$, $(2,1)$, $(2,2)$, $(0,4)$, and $(5,5)$. On the set of edges E, a weight function $c : E \to \mathbf{R}$ is defined as follows: $c(e)$ is equal to the Euclidean distance between the vertices connected by edge e, $e \in E$. Using Kruskal's method, determine the minimum spanning tree for this graph, marking the edges selected by this method step by step.

10. In the complete undirected network (V, E, c) with n vertices, it is required to find the shortest Hamiltonian cycle. Which of the following statements are true:

(a) This problem is the symmetric Traveling Salesman Problem.
(b) This problem is a shortest path problem.
(c) This problem can be solved using Ford's algorithm.
(d) The 2-optimal algorithm applied to this problem always gives an optimal solution.
(e) The problem can be solved most quickly by searching all Hamiltonian cycles.
(f) This problem has n^{n-2} feasible solutions.
(g) This problem has exponential computational complexity.

11. An undirected graph $G = (V, E)$ is given with a distinguished source r and sink s, on which a capacity matrix U is defined with non-negative integer elements.

(a) Define the concept of a feasible (r, s)-flow in this network.
(b) Define the maximum flow problem in this network.
(c) What is the total flow in the network?
(d) Does this problem always have a feasible solution? Justify your answer.
(e) Does this problem always have an optimal solution? Justify your answer.
(f) Can it be assumed without loss of generality that the graph $G = (V, E)$ is complete for the maximum flow problem? Justify your answer.

12. A directed graph (V, E, c) is given with the weight matrix:

$$C = \begin{bmatrix} 15 & 11 & 6 & 7 & 9 \\ 7 & 4 & 8 & 9 & 13 \\ 6 & 9 & 12 & 5 & 7 \\ 5 & 9 & 3 & 14 & 12 \\ 7 & 10 & 6 & 5 & 8 \end{bmatrix}$$

For the Traveling Salesman Problem on this graph, perform one iteration of the branch-and-bound method using the reduction method to determine the lower bound:

(a) Compute the reduced weight matrix.
(b) Determine the lower bound for the length of the shortest Hamiltonian cycle in this graph for the set $D_{\emptyset,\emptyset}$.
(c) According to which edge should the division occur?

13. For the maximum flow problem in the undirected graph $G = (V, E)$ with the source r at vertex 1 and sink s at vertex 4, the capacity matrix is given as:

$$U = \begin{bmatrix} 0 & 5 & 4 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Draw this graph, indicating the capacities.
(b) Provide an example of a feasible (r, s)-flow.
(c) Present the residual network and the augmenting path.