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Operations research tasks – list 2

1. Let A_i be a matrix of type $m \times n_i, i = 1, ..., m$, and B_i –a matrix of type $n_i \times p, i = 1, ..., m$. Show that
$$\begin{bmatrix} A_1 & \cdots & A_m \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix} = \sum_{i=1}^{m} A_i B_i,$$
that is, the rule for block matrix multiplication is the same as the rule for 'ordinary' matrix multiplication.

2. Let A be a matrix of type $m \times n$, $x = (x_1, ..., x_n)^\top$ and $y = (y_1, ...y_m)^\top$. Show that Ax is a linear combination of the columns of matrix A with coefficients $x_1, ..., x_n$ and $y^\top A$ is a linear combination of the columns of matrix A with coefficients $y_1, ..., y_m$.

3. Let A be a matrix of type $m \times n$. Show that $Ae_j = A_j$ i $e_i^\top A = A^i$, where e_k denotes the kth unit vector of the Euclidean space of the appropriate dimension, A_j denotes the j-th column of A and A^i denotes the ith row of A.

4. Let A be a matrix of type $m \times n$. Show that $A^\top A$ is non-negative definite and that it is positive definite if and only if A is of full column rank.

5. Given a homogeneous system of m linear equations with n unknowns $Ax = 0$.
   (a) Show that the solution set of this system is a linear subspace.
   (b) Suppose that the first m columns of the matrix A are linearly independent. Let us denote the submatrix formed from these columns with the symbol A_B, and the matrix formed with the remaining $n-m$ columns with the symbol A_N. Present the form of any solution of this system.
   (c) What is the basis and dimension of the linear subspace in question?

6. Given a consistent system of m linear equations $Ax = b$.
   (a) Explain why, without loss of generality, it may be assumed that matrix A has full row row, i.e. that $\mathrm{r}(A) = m$.
   (b) Show that the solution set of this system is an affine subspace, i.e

   if x, y are solutions of the system $Ax = b$ and $\lambda \in \mathbb{R}$,
   then $(1-\lambda)x + \lambda y$ is also a solution of this system.

   (c) Present the form of any solution of this system.

7. Solve the following systems of equations using Gaussian elimination:

$$\text{(a)} \begin{array}{r} x_1 + x_2 - x_3 = 2 \\ -2x_1 + x_2 + x_3 = 3 \\ x_1 + x_2 + x_3 = 6 \end{array}, \quad \text{(b)} \begin{array}{r} x_1 + x_2 - x_3 = 2 \\ -2x_1 + x_2 + x_3 = 3 \\ 3x_1 - 2x_3 = -1 \end{array}, \quad \text{(c)} \begin{array}{r} x_1 + x_2 - x_3 = 2 \\ -2x_1 + x_2 + x_3 = 3 \\ 3x_1 - 2x_3 = 1 \end{array},$$

$$\text{(d)} \begin{array}{r} 2x_1 + x_2 - x_3 + x_4 = 1 \\ x_1 + 3x_2 - 3x_3 = 1 \\ x_1 + x_2 + x_3 - x_4 = 1 \end{array}, \quad \text{(e)} \begin{array}{r} x_1 + 2x_2 - x_3 - x_4 = 1 \\ x_1 + x_2 + x_3 + 3x_4 = 2 \\ 3x_1 + 5x_2 - x_3 + x_4 = 3 \end{array}, \quad \text{(f)} \begin{array}{r} 2x_1 + x_2 + x_3 = 1 \\ 3x_1 - x_2 + 3x_3 = 2 \\ x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 = 1 \end{array}$$

8. Let $c_1 \in \mathbb{R}^{n_1}$, $c_2 \in \mathbb{R}^{n_2}$, $b_1 \in \mathbb{R}^{m_1}$, $b_2 \in \mathbb{R}^{m_2}$ and A_{11}, A_{12}, A_{21}, A_{22} be matrices of appropriate types> Consider the following linear programming problem:

$$\begin{array}{rr} \text{maximiza} & c_1^\top x_1 + c_2^\top x_2 \\ \text{with respect to} & (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \\ \text{subject to} & A_{11}x_1 + A_{12}x_2 = b_1 \\ & A_{21}x_1 + A_{22}x_2 \le b_2 \\ & x_1 \ge 0. \end{array}$$

Present this task in classic and standard form.

9. Represent the following tasks as linear programming problem:

(a) (linear feasibility problem)

$$\text{determine the solution to the system of linear inequalities } a_i^\intercal x \le b_i, i = 1, ..., m,$$
$$\text{gdzie } a_i = (a_{i1}, ..., a_{in})^\intercal \in \mathbb{R}^n, i = 1, ..., m, x = (x_1, ..., x_n)^\intercal \in \mathbb{R}^n.$$

(b) (non-differentiable minimization problem)

$$\text{minimize } f(x) = \max\{\ a_i^\intercal x - b_i, i = 1, ..., m\},$$
$$\text{with respect to } x \in \mathbb{R}^n.$$

Hint: introduce an auxiliary variable u and note that $f(x) \le u$ if and only if $f_i(x) := a_i^\intercal x - b_i \le u$, $i = 1, ..., m$.

(c)

$$\text{minimize } f(x) = \sum_{i=1}^m |a_i^\intercal x - b_1|\,,$$
$$\text{with respect to } x \in \mathbb{R}^n.$$