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Operations research tasks – list 3

1. Check whether the following subsets are convex:
   (a) polyhedral set (cross-section of finitely many half-spaces),
   (b) set of possible solutions to a linear programming problem,
   (c) a set of optimal solutions to a linear programming problem,
   (d) a set of basic possible solutions to a linear programming problem in standard form,
   (e) set of solutions to a system of linear equations,
   (f) budget set,
   (g) set of all possible share portfolios on the Warsaw Stock Exchange,
   (h) the ball $B(y,r) = \{x \in \mathbb{R}^n : \|x - y\| \leq r\}$,
   (i) the sublevel $S(f,\alpha) = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ of a convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$,
   (j) the superlevel $S^+(f,\alpha) = \{x \in \mathbb{R}^n : f(x) \geq \alpha\}$ of a concave function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.
2. Give examples of other convex sets occurring in economic problems.
3. Determine the vertices of:
   (a) the standard simplex $\Delta_n = \{x \in \mathbb{R}^n : x \geq 0, e^\top x = 1\}$,
   (b) set of solutions to the system of linear equations:
      1. tha has a unique solution,
      2. that has more solutions.
4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing convex function. Show that the function $g \circ f$ is convex.
5. Let $h : \mathbb{R}^m \to \mathbb{R}$ be a convex function and A be a matrix of type $m \times n$. Show that the function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = h(Ax)$ is convex.
6. Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., m$ be a convex functions and $f = \max_i f_i$. Show that the function f is convex.
7. Show that the following functions are convex:
   (a) a linear function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = c^\top x$, gdzie $c \in \mathbb{R}^n$,
   (b) the exponential function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2^x$,
   (c) the power function $f : \mathbb{R}_+ \to \mathbb{R}$, $f(x) = x^\alpha$ for $\alpha \geq 1$,
   (d) arbitrary norm $\|x\|$ in $\mathbb{R}^n$,
   (e) a function that is the square of any norm in $\mathbb{R}^n$, i.e. the function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|x\|^2$. **Hint**: use the task 4.
8. The following utility functions are given $f : \mathbb{R}^2_+ \to \mathbb{R}$:
   (a) $f(x_1, x_2) = 2x_1 + 3x_2$,
   (b) $f(x_1, x_2) = \min\{x_1, x_2\}$,
   (c) $f(x_1, x_2) = \sqrt{x_1 x_2}$.

   Draw contour lines for these functions. Are the superlevels of these functions convex sets? Are these functions concave? **Hint**: In example (c), determine the matrix of second-order partial derivatives and check its definiteness.
9. Find the relationship between income tax and income in accordance with the applicable Personal Income Tax Act. Is it a convex function. Explain why it is beneficial to tax spouses jointly. Is it also beneficial if one of the spouses receives income from abroad?
10. Give examples of other convex functions and concave functions occurring in economic or technical problems.