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ON STOCHASTIC FRACTIONAL VOLTERRA EQUATIONS IN HILBERT SPACE

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Abstract. In this paper, stochastic Volterra equations, particularly fractional, in Hilbert space are studied. Sufficient conditions for existence of strong solutions are provided.

1. **Introduction.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis and H a separable Hilbert space. In this paper we consider the stochastic Volterra equations in H of the form

$$X(t) = X(0) + \int_0^t a(t-\tau)AX(\tau)d\tau + \int_0^t \Psi(\tau)\, dW(\tau), \quad t \geq 0\,. \tag{1}$$

In (1), $X(0)$ is an H-valued $\mathcal{F}_0$-measurable random variable and $a \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ is a scalar kernel. The operator A is closed linear unbounded in H with a dense domain $D(A)$ equipped with the graph norm $|\cdot|_{D(A)}$, i.e. $|h|_{D(A)} := (|h|_H^2 + |Ah|_H^2)^{1/2}$, where $|\cdot|_H$ denotes the norm in H. W is a cylindrical Wiener process (see e.g. [3] or [7] for the definition, properties and the stochastic integral with respect to that process) on another separable Hilbert space U, with the covariance operator $Q \in L(U)$. Q is a linear symmetric positive operator with $\mathrm{Tr}\, Q = +\infty$ and Ψ is an appropriate process defined below.

Equations (1) contain important special cases, e.g. heat, wave and integro-differential equations. Moreover, (1) are motivated by a wide class of model problems and correspond to abstract stochastic versions of several deterministic problems, mentioned, e.g. in [13] (see also the references therein).

In order to provide a sense for the integral $\int_0^t \Psi(\tau)dW(\tau)$, the process $\Psi(t)$, $t \geq 0$, has to be an operator-valued process (see, e.g. [7]). We define the subspace $U_0 := Q^{1/2}(U)$ of the space U endowed with the inner product $\langle u, v\rangle_{U_0} := \langle Q^{-1/2}u, Q^{-1/2}v\rangle_U$. By $L_2^0 := L_2(U_0, H)$ we denote the set of all Hilbert-Schmidt operators acting from U_0 into H; the set L_2^0 equipped with the norm $||C||_{L_2(U_0,H)} := \left(\sum_{k=1}^{+\infty} |Cu_k|_H^2\right)^{\frac{1}{2}}$, is a separable Hilbert space.

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By $\mathcal{N}^2(0,T;L_2^0)$, where $T < +\infty$ is fixed, we denote a Hilbert space of all L_2^0-predictable processes Ψ such that $||\Psi||_T < +\infty$, where

$$||\Psi||_T := \left\{ \mathbb{E}\left(\int_0^T ||\Psi(\tau)||^2_{L_2^0}\, d\tau \right) \right\}^{\frac{1}{2}} = \left\{ \mathbb{E} \int_0^T \left[\mathrm{Tr}(\Psi(\tau)Q^{\frac{1}{2}})(\Psi(\tau)Q^{\frac{1}{2}})^* \right] d\tau \right\}^{\frac{1}{2}} .$$

If $\Psi \in \mathcal{N}^2(0,T;L_2^0)$, then the integral $\int_0^t \Psi(\tau)\, dW(\tau)$ makes sense.

Let us note that the results obtained below for cylindrical Wiener process ($\mathrm{Tr}\, Q = +\infty$) hold for genuine Wiener process ($\mathrm{Tr}\, Q < +\infty$), too. In the latter case, that is, if Q is a nuclear operator, $L(U,H) \subset L_2(U_0,H)$ and then the stochastic integral $\int_0^t \Psi(\tau)dW(\tau)$ is well defined (for details, see [7]).

In this paper, we use the so-called resolvent approach to the Volterra equation (1) (for details we refer to [13]).

First, we recall some definitions connected with deterministic version of (1), that is, the equation

$$u(t) = \int_0^t a(t-\tau)\, Au(\tau)d\tau + f(t), \quad t \geq 0, \tag{2}$$

where f is an H-valued function. In (2), the kernel function $a(t)$ and the operator A are the same like previously.

Definition 1. A family $(S(t))_{t\geq 0}$ of bounded linear operators in H is called `resolvent` for (2) if the following conditions are satisfied:

1. $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;
2. $S(t)$ commutes with the operator A:
   $S(t)(D(A)) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;
3. the following `resolvent equation` holds
   $$S(t)x = x + \int_0^t a(t-\tau)A\, S(\tau)x\, d\tau \tag{3}$$
   for all $x \in D(A),\ t \geq 0$.

We will assume in the sequel that the resolvent family $(S(t))_{t\geq 0}$, to (2) exists.

Let us emphasize that the family $(S(t))_{t\geq 0}$ does not create in general any semigroup and that $S(t),\ t \geq 0$, are generated by the pair $(A, a(t))$, that is, the operator A and the kernel function $a(t),\ t \geq 0$.

A consequence of the strong continuity of $S(t)$ is that $\sup_{t\leq T}\ ||S(t)|| < +\infty$ for any $T \geq 0$.

Definition 2. We say that the function $a \in L^1(0,T)$ is `completely positive` on $[0,T]$, if for any $\mu \geq 0$, the solutions of the equations

$$s(t) + \mu(a\star s)(t) = 1 \quad \text{and} \quad r(t) + \mu(a \star r)(t) = a(t) \tag{4}$$

satisfy $s(t) \geq 0$ and $r(t) \geq 0$ on $[0,T]$.

The class of completely positive kernels, introduced in [2], arise naturally in applications, see [13]. For instance, the functions $a(t) \equiv 1$, $a(t) = t$, $a(t) = e^{-t}$, $t \geq 0$, are completely positive.

Definition 3. Suppose $S(t),\ t \geq 0$, is a resolvent. $S(t)$ is called `exponentially bounded` if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$||S(t)|| \leq M\, e^{\omega t}, \text{ for all } t \geq 0\ ;$$

(M, ω) is called a `type` of $S(t)$.

Let us note that contrary to C_0-semigroups, not every resolvent family needs to be exponentially bounded; for counterexamples we refer to [4].

In the paper, the key role is played by the following, not yet published, result providing a convergence of resolvents.

Theorem 1. *Let A be the generator of a C_0-semigroup in H and suppose the kernel function a is completely positive. Then (A, a) admits an exponentially bounded resolvent $S(t)$. Moreover, there exist bounded operators A_n such that (A_n, a) admit resolvent families $S_n(t)$ satisfying $||S_n(t)|| \leq Me^{w_0 t}$ $(M \geq 1,\ w_0 \geq 0)$ for all $t \geq 0,\ n \in \mathbb{N}$, and*

$$S_n(t)x \to S(t)x \quad \text{as} \quad n \to +\infty \tag{5}$$

for all $x \in H,\ t \geq 0$.

Additionally, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

Proof. The first assertion follows directly from [12, Theorem 5] (see also [13, Theorem 4.2]). Since A generates a C_0-semigroup $T(t),\ t \geq 0$, the resolvent set $\rho(A)$ of A contains the ray $[w, \infty)$ and

$$||R(\lambda, A)^k|| \leq \frac{M}{(\lambda - w)^k} \qquad \text{for } \lambda > w, \qquad k \in \mathbb{N},$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$.

Define

$$A_n := nAR(n, A) = n^2 R(n, A) - nI, \qquad n > w \tag{6}$$

the `Yosida approximation` of A, where $R(n, A) = (nI - A)^{-1}$. For details, see e.g. [11].

Then

$$\begin{aligned} ||e^{tA_n}|| &= e^{-nt}||e^{n^2 R(n,A)t}|| \leq e^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k} t^k}{k!} ||R(n, A)^k|| \\ &\leq Me^{(-n + \frac{n^2}{n-w})t} = Me^{\frac{nwt}{n-w}}. \end{aligned}$$

Hence, for $n > 2w$ we obtain

$$||e^{A_n t}|| \leq Me^{2wt}. \tag{7}$$

Taking into account the above estimate and the complete positivity of the kernel function a, we can follow the same steps as in [12, Theorem 5] to obtain that there exist constants $M_1 > 0$ and $w_1 \in \mathbb{R}$ (independent of n, due to (7)) such that

$$||[H_n(\lambda)]^{(k)}|| \leq \frac{M_1}{(\lambda - w_1)^{k+1}} \qquad \text{for } \lambda > w_1,$$

where $H_n(\lambda) := (\lambda - \lambda \hat{a}(\lambda) A_n)^{-1}$. Here and in the sequel the hat indicates the Laplace transform. Hence, the generation theorem for resolvent families implies that for each $n > 2\omega$, the pair (A_n, a) admits resolvent family $S_n(t)$ such that

$$||S_n(t)|| \leq M_1 e^{w_1 t}. \tag{8}$$

In particular, the Laplace transform $\hat{S}_n(\lambda)$ exists and satisfies

$$\hat{S}_n(\lambda) = H_n(\lambda) = \int_0^\infty e^{-\lambda t} S_n(t) dt, \qquad \lambda > w_1.$$

Now recall from semigroup theory that for all μ sufficiently large we have

$$R(\mu, A_n) = \int_0^\infty e^{-\mu t}\, e^{A_n t}\, dt$$

as well as,

$$R(\mu, A) = \int_0^\infty e^{-\mu t}\, T(t)\, dt\, .$$

Since $\hat{a}(\lambda) \to 0$ as $\lambda \to \infty$, we deduce that for all λ sufficiently large, we have

$$H_n(\lambda) := \frac{1}{\lambda \hat{a}(\lambda)} R(\frac{1}{\hat{a}(\lambda)}, A_n) = \frac{1}{\lambda \hat{a}(\lambda)} \int_0^\infty e^{(-1/\hat{a}(\lambda))t} e^{A_n t} dt\, ,$$

and

$$H(\lambda) := \frac{1}{\lambda \hat{a}(\lambda)} R(\frac{1}{\hat{a}(\lambda)}, A) = \frac{1}{\lambda \hat{a}(\lambda)} \int_0^\infty e^{(-1/\hat{a}(\lambda))t} T(t) dt\, .$$

Hence, from the identity

$$H_n(\lambda) - H(\lambda) = \frac{1}{\lambda \hat{a}(\lambda)} [R(\frac{1}{\hat{a}(\lambda)}, A_n) - R(\frac{1}{\hat{a}(\lambda)}, A)]$$

and the fact that $R(\mu, A_n) \to R(\mu, A)$ as $n \to \infty$ for all μ sufficiently large (see, e.g. [11, Lemma 7.3], we obtain that

$$H_n(\lambda) \to H(\lambda) \quad \text{as } n \to \infty\, . \tag{9}$$

Finally, due to (8) and (9) we can use the Trotter-Kato theorem for resolvent families of operators (cf. [9, Theorem 2.1]) and the conclusion follows. □

Remark 1. (a) The convergence (5) is an extension of the result due to Clément and Nohel [2].

(b) The above theorem gives a partial answer to the following open problem for a resolvent family $S(t)$ generated by a pair (A, a): do there exist bounded linear operators A_n generating resolvent families $S_n(t)$ such that $S_n(t)x \to S(t)x$?. In particular case $a(t) \equiv 1$, A_n are provided by the Hille-Yosida approximation of A and additionally $S_n(t) = e^{tA_n}$.

2. **Probabilistic results.** In the sequel we shall use the following `Probability Assumptions`, abbr. (PA):

1. $X(0)$ is an H-valued, $\mathcal{F}_0$-measurable random variable;
2. $\Psi \in \mathcal{N}^2(0, T; L_2^0)$ and the interval $[0, T]$ is fixed.

The following types of definitions of solutions to (1) are possible, see [8].

Definition 4. Assume that (PA) hold. An H-valued predictable process $X(t)$, $t \in [0, T]$, is said to be a **`strong solution`** to (1), if X has a version such that $P(X(t) \in D(A)) = 1$ for almost all $t \in [0, T]$; for any $t \in [0, T]$

$$\int_0^t |a(t-\tau) A X(\tau)|_H\, d\tau < +\infty, \quad P\text{-a.s.} \tag{10}$$

and for any $t \in [0, T]$ the equation (1) holds P-a.s.

Let A^* be the adjoint of A with a dense domain $D(A^*) \subset H$ and the graph norm $|\cdot|_{D(A^*)}$ defined as follows: $|h|_{D(A^*)} := (|h|_H^2 + |A^* h|_H^2)^{1/2}$ for $h \in D(A^*)$.

Definition 5. Let (PA) hold. An H-valued predictable process $X(t),\ t\in[0,T]$, is said to be a `weak solution` to (1), if $P(\int_0^t |a(t-\tau)X(\tau)|_H d\tau < +\infty) = 1$ and if for all $\xi\in D(A^*)$ and all $t\in[0,T]$ the following equation holds

$$\begin{aligned}\langle X(t),\xi\rangle_H =& \langle X(0),\xi\rangle_H + \langle \int_0^t a(t-\tau)X(\tau)\,d\tau, A^*\xi\rangle_H \\ &+ \langle \int_0^t \Psi(\tau)dW(\tau),\xi\rangle_H, \quad P\text{−a.s.}\end{aligned}$$

Definition 6. Assume that $X(0)$ is $\mathcal{F}_0$-measurable random variable. An H-valued predictable process $X(t),\ t\in[0,T]$, is said to be a `mild solution` to the stochastic Volterra equation (1), if $\mathbb{E}(\int_0^t ||S(t-\tau)\Psi(\tau)||^2_{L_2^0}\,d\tau) < +\infty$ for $t\le T$ and, for arbitrary $t\in[0,T]$,

$$X(t) = S(t)X(0) + \int_0^t S(t-\tau)\Psi(\tau)\,dW(\tau), \quad P\text{−a.s.} \tag{11}$$

The integral appearing in (11) will be called `stochastic convolution` and denoted by

$$W^\Psi(t) := \int_0^t S(t-\tau)\Psi(\tau)\,dW(\tau), \qquad t\ge 0, \tag{12}$$

where $\Psi\in\mathcal{N}^2(0,T;L_2^0)$.

We will show in the sequel that the convolution W^Ψ is a weak solution to (1) and next we will provide sufficient conditions under which W^Ψ is a strong solution to (1), as well.

Let us recall (from [3] and [8]) some properties of the convolution $W^\Psi(t)$, $t\ge 0$.

Proposition 1. *(see, e.g.*[3, Proposition 4.15]*)*
Assume that A is a closed linear unbounded operator with the dense domain $D(A)\subset H$ and $\Phi(t),\ t\in[0,T]$ is an $L_2(U_0,H)$-predictable process. If $\Phi(t)(U_0)\subset D(A)$, $P-a.s.$ for all $t\in[0,T]$ and

$$P\left(\int_0^T ||\Phi(s)||^2_{L_2^0}\,ds<\infty\right)=1,\ P\left(\int_0^T ||A\Phi(s)||^2_{L_2^0}\,ds<\infty\right)=1,$$

$$\text{then} \qquad P\left(\int_0^T \Phi(s)\,dW(s)\in D(A)\right)=1$$

$$\text{and} \qquad A\int_0^T \Phi(s)\,dW(s) = \int_0^T A\Phi(s)\,dW(s),\ P-a.s.$$

For the proofs of Propositions 2, 3 and 4 we refer to [8].

Proposition 2. *Assume that (2) admits resolvent operators $S(t),\ t\ge 0$. Then, for arbitrary process $\Psi\in\mathcal{N}^2(0,T;L_2^0)$, the process $W^\Psi(t),\ t\ge 0$, given by (12) has a predictable version.*

Proposition 3. *Assume that $\Psi\in\mathcal{N}^2(0,T;L_2^0)$. Then the process $W^\Psi(t),\ t\ge 0$, defined by (12) has square integrable trajectories.*

Proposition 4. *If $\Psi \in \mathcal{N}^2(0,T;L_2^0)$, then the stochastic convolution W^Ψ fulfills the equation*

$$\langle W^\Psi(t),\xi\rangle_H = \int_0^t \langle a(t-\tau)W^\Psi(\tau), A^*\xi\rangle_H + \int_0^t \langle \xi, \Psi(\tau)dW(\tau)\rangle_H, \quad P-a.s.$$

for any $t\in[0,T]$ and $\xi \in D(A^*)$.

Proposition 4 shows that the convolution W^Ψ is a weak solution to (1) (see [8]) and enables us to formulate the following results.

Proposition 5. *Let A be the generator of C_0-semigroup in H and suppose that the function a is completely positive. If Ψ and $A\Psi$ belong to $\mathcal{N}^2(0,T;L_2^0)$ and in addition $\Psi(t)(U_0)\subset D(A)$, P-a.s., then the following equality holds*

$$W^\Psi(t) = \int_0^t a(t-\tau)A\,W^\Psi(\tau)\,d\tau + \int_0^t \Psi(\tau)\,dW(\tau), \quad P-a.s. \tag{13}$$

Proof. Because formula (13) holds for any bounded operator, then it holds for the Yosida approximation A_n of the operator A, too, that is

$$W_n^\Psi(t) = \int_0^t a(t-\tau)A_nW_n^\Psi(\tau)d\tau + \int_0^t \Psi(\tau)dW(\tau),$$

where

$$W_n^\Psi(t) := \int_0^t S_n(t-\tau)\Psi(\tau)dW(\tau)$$

and

$$A_nW_n^\Psi(t) = A_n\int_0^t S_n(t-\tau)\Psi(\tau)dW(\tau).$$

Recall that by assumption $\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Because the operators $S_n(t)$ are deterministic and bounded for any $t\in[0,T]$, $n\in\mathbb{N}$, then the operators $S_n(t-\cdot)\Psi(\cdot)$ belong to $\mathcal{N}^2(0,T;L_2^0)$, too. In consequence, the difference

$$\Phi_n(t-\cdot) := S_n(t-\cdot)\Psi(\cdot) - S(t-\cdot)\Psi(\cdot) \tag{14}$$

belongs to $\mathcal{N}^2(0,T;L_2^0)$ for any $t\in[0,T]$ and $n\in\mathbb{N}$. This means that

$$\mathbb{E}\left(\int_0^t |\Phi_n(t-\tau)|^2_{L_2^0}d\tau\right) < +\infty \tag{15}$$

for any $t\in[0,T]$.

Let us recall (see [7]) that the cylindrical Wiener process $W(t)$, $t\geq 0$, can be written in the form

$$W(t) = \sum_{j=1}^{+\infty} f_j\,\beta_j(t), \tag{16}$$

where $\{f_j\}$ is an orthonormal basis of U_0 and $\beta_j(t)$ are independent real Wiener processes. From (16) we have

$$\int_0^t \Phi_n(t-\tau)\,dW(\tau) = \sum_{j=1}^{+\infty}\int_0^t \Phi_n(t-\tau)\,f_j\,d\beta_j(\tau). \tag{17}$$

Then, from (15)

$$\mathbb{E}\left[\int_0^t \left(\sum_{j=1}^{+\infty} |\Phi_n(t-\tau)\,f_j|_H^2\right)d\tau\right] < +\infty \tag{18}$$

for any $t \in [0,T]$. Next, from (17), properties of stochastic integral and (18) we obtain for any $t \in [0,T]$, that

$$\begin{aligned} \mathbb{E}\left|\int_0^t \Phi_n(t-\tau)\, dW(\tau)\right|_H^2 &= \mathbb{E}\left|\sum_{j=1}^{+\infty}\int_0^t \Phi_n(t-\tau)\, f_j\, d\beta_j(\tau)\right|_H^2 \leq \\ \mathbb{E}\left[\sum_{j=1}^{+\infty}\int_0^t |\Phi_n(t-\tau)\, f_j|_H^2 d\tau\right] &\leq \mathbb{E}\left[\sum_{j=1}^{+\infty}\int_0^T |\Phi_n(T-\tau)\, f_j|_H^2 d\tau\right] < +\infty. \end{aligned}$$

By Theorem 1, the convergence (5) of resolvent families is uniform in t on every compact subset of $\mathbb{R}_+$, particularly on the interval $[0,T]$. Then, for any fixed j,

$$\int_0^T |[S_n(T-\tau) - S(T-\tau)]\, \Psi(\tau)\, f_j|_H^2 d\tau \longrightarrow 0, \quad \text{as} \quad n \to infty. \tag{19}$$

So, using (18) and (19) we can write

$$\begin{aligned} \sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t \Phi_n(t-\tau)dW(\tau)\right|_H^2 &\equiv \sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t [S_n(t-\tau)-S(t-\tau)]\Psi(\tau)dW(\tau)\right|_H^2 \\ &\leq \mathbb{E}\left[\sum_{j=1}^{+\infty}\int_0^T |[S_n(T-\tau)-S(T-\tau)]\Psi(\tau)\, f_j|_H^2 d\tau\right] \to 0 \end{aligned}$$

as $n \to +\infty$.

Hence, by the Lebesgue dominated convergence theorem

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}\left|W_n^\Psi(t) - W^\Psi(t)\right|_H^2 = 0. \tag{20}$$

By assumption, $\Psi(t)(U_0) \subset D(A)$, $P - a.s.$ Because $S(t)(D(A)) \subset D(A)$, then $S(t-\tau)\Psi(\tau)(U_0) \subset D(A)$, $P - a.s.$, for any $\tau \in [0,t]$, $t \geq 0$. Hence, by Proposition 1, $P(W^\Psi(t) \in D(A)) = 1$.

For any $n \in \mathbb{N}$, $t \geq 0$, we can estimate

$$|A_n W_n^\Psi(t) - AW^\Psi(t)|_H^2 < 3[N_{n,1}^2(t) + N_{n,2}^2(t)], \tag{21}$$

where

$$\begin{aligned} N_{n,1}(t) &:= |A_n W_n^\Psi(t) - A_n W^\Psi(t)|_H, \\ N_{n,2}(t) &:= |A_n W^\Psi(t) - AW^\Psi(t)|_H\, . \end{aligned}$$

Using the convergence of resolvents (5) and the Yoshida approximation properties, we can follow the same steps as above for proving

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}(N_{n,1}^2(t)) \to 0$$

and

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}(N_{n,2}^2(t)) \to 0\, .$$

Therefore, we can deduce that

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}|A_n W_n^\Psi(t) - AW^\Psi(t)|_H^2 = 0,$$

and then (13) holds. □

Theorem 2. *Suppose that assumptions of Proposition 5 hold. Then the equation (1) has a strong solution. Precisely, the convolution W^Ψ given by (12) is the strong solution to (1).*

Proof. In order to prove Theorem 2, we have to show only the condition (10). Let us note that the convolution W^Ψ has integrable trajectories. Because the closed unbounded linear operator A becomes bounded on $(D(A), |\cdot|_{D(A)})$, see [14, Chapter 5], we obtain that $AW^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. Hence, properties of convolution provide integrability of the function $a(T-\tau)AW^\Psi(\tau)$ with respect to τ, what finishes the proof. □

3. **Fractional Volterra equations.** As we have already written, (2) contains some class of equations. For instance when $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\alpha > 0$, we obtain integro-differential equations studied by many authors, see e.g. [1] and references therein. These facts lead us to the fractional stochastic Volterra equations of the form

$$X(t) = X(0) + \int_0^t g_\alpha(t-\tau)AX(\tau)d\tau + \int_0^t \Psi(\tau)\, dW(\tau), \quad t \geq 0, \tag{22}$$

where $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\alpha > 0$. Let us emphasize that for $\alpha \in (0,1]$, g_α are completely positive, but for $\alpha > 1$, g_α are not completely positive.

Now, the pairs $(A, g_\alpha(t))$ generate α-times resolvents $S_\alpha(t),\ t \geq 0$ which are analogous to resolvents defined in section 1; for more details, see [1].

Remark 2. Observe that the α-times resolvent family corresponds to a C_0-semigroup in case $\alpha = 1$ and a cosine family in case $\alpha = 2$. (Let us recall, e.g. from [5], that a family $\mathcal{C}(t)$, $t \geq 0$, of linear bounded operators on H is called `cosine family` if for every $t, s \geq 0$, $t > s$: $\mathcal{C}(t+s) + \mathcal{C}(t-s) = 2\mathcal{C}(t)\mathcal{C}(s)$.) In consequence, when $1 < \alpha < 2$ such resolvent families interpolate C_0-semigroups and cosine functions. In particular, for $A = \Delta$, the integro-differential equations corresponding to such resolvent families interpolate the heat equation and the wave equation, see, e.g. [6].

We consider two cases:

(A1): A is the generator of C_0-semigroup and $\alpha \in (0,1)$;

(A2): A is the generator of a strongly continuous cosine family and $\alpha \in (0,2)$.

In this part of the paper, the results concerning a weak convergence of α-times resolvents play the key role. Using the very recent result due to Li and Zheng [10], we can formulate the approximation theorems for fractional Volterra equations.

Theorem 3. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ in H such that $||T(t)|| \leq Me^{\omega t},\ t \geq 0$. Then, for each $0 < \alpha < 1$ there exist bounded operators A_n and α-times resolvent families $S_{\alpha,n}(t)$ for A_n satisfying $||S_{\alpha,n}(t)|| \leq MCe^{(2\omega)^{1/\alpha}t}$, for all $t \geq 0,\ n \in \mathbb{N}$, and*

$$S_{\alpha,n}(t)x \to S_\alpha(t)x \quad as \quad n \to +\infty \tag{23}$$

for all $x \in H,\ t \geq 0$. Moreover, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

Outline of the proof. The first assertion follows from [1, Theorem 3.1], that is, for each $0 < \alpha < 1$ there is an α-times resolvent family $(S_\alpha(t))_{t\geq 0}$ for A given by

$$S_\alpha(t)x = \int_0^\infty \varphi_{t,\alpha}(s)T(s)xds, \quad t > 0,$$

where $\varphi_{t,\gamma}(s) := t^{-\gamma}\Phi_\gamma(st^{-\gamma})$ and $\Phi_\gamma(z)$ is the Wright function defined as

$$\Phi_\gamma(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1.$$

Define

$$A_n := nAR(n,A) = n^2 R(n,A) - nI, \qquad n > w,$$

the *Yosida approximation* of A.

Since each A_n is bounded, it follows that for each $0 < \alpha < 1$ there exists an α-times resolvent family $(S_{\alpha,n}(t))_{t\geq 0}$ for A_n given as

$$S_{\alpha,n}(t) = \int_0^\infty \varphi_{t,\alpha}(s) e^{sA_n}\, ds, \quad t > 0.$$

We recall that the Laplace transform of the Wright function corresponds to $E_\gamma(-z)$ where E_γ denotes the Mittag-Leffler function. In particular, $\Phi_\gamma(z)$ is a probability density function. It follows that for $t \geq 0$:

$$\begin{aligned} \|S_{\alpha,n}(t)\| &\leq \int_0^\infty \varphi_{t,\alpha}(s) \|e^{sA_n}\| ds \\ &\leq M \int_0^\infty \varphi_{t,\alpha}(s) e^{2\omega s} ds = M \int_0^\infty \Phi_\alpha(\tau) e^{2\omega t^\alpha \tau} d\tau = M E_\alpha(2\omega t^\alpha). \end{aligned}$$

The continuity in $t \geq 0$ of the Mittag-Leffler function and its asymptotic behavior, imply that for $\omega \geq 0$ there exists a constant $C > 0$ such that

$$E_\alpha(\omega t^\alpha) \leq C e^{\omega^{1/\alpha} t}, \quad t \geq 0,\ \alpha \in (0,2).$$

This gives

$$\|S_{\alpha,n}(t)\| \leq MCe^{(2\omega)^{1/\alpha} t}, \quad t \geq 0.$$

Now we recall the fact that $R(\lambda, A_n)x \to R(\lambda, A)x$ as $n \to \infty$ for all λ sufficiently large (e.g. [11, Lemma 7.3]), so we can conclude from [10, Theorem 4.2] that

$$S_{\alpha,n}(t)x \to S_\alpha(t)x \quad \text{as} \quad n \to +\infty$$

for all $x \in H$, uniformly for t on every compact subset of $\mathbb{R}_+$ □

An analogous convergence for α-times resolvents can be proved in another case, too.

Theorem 4. *Let A be the generator of a C_0-cosine family $(T(t))_{t\geq 0}$ in H. Then, for each $0 < \alpha < 2$ there exist bounded operators A_n and α-times resolvent families $S_{\alpha,n}(t)$ for A_n satisfying $||S_{\alpha,n}(t)|| \leq MCe^{(2\omega)^{1/\alpha}t}$, for all $t \geq 0$, $n \in \mathbb{N}$, and $S_{\alpha,n}(t)x \to S_\alpha(t)x$ as $n \to +\infty$ for all $x \in H$, $t \geq 0$. Moreover, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.*

Now, we are able to formulate the result analogous to that in section 2.

Theorem 5. *Assume that* **(A1)** *or* **(A2)** *holds. If Ψ and $A\Psi$ belong to $\mathcal{N}^2(0,T;L_2^0)$ and in addition $\Psi(t)(U_0) \subset D(A)$, P-a.s., then the equation (1) has a strong solution. Precisely, the convolution*

$$W_\alpha^\Psi(t) := \int_0^t S_\alpha(t-\tau)\,\Psi(\tau)\,dW(\tau)$$

is the strong solution to (1).

Outline of the proof. First, analogously like in section 2, we show that the convolution $W_\alpha^\Psi(t)$ fulfills the following equation

$$W_\alpha^\Psi(t) = \int_0^t g_\alpha(t-\tau) A\, W_\alpha^\Psi(\tau)\, d\tau + \int_0^t \Psi(\tau)\, dW(\tau)\,. \tag{24}$$

Next, we have to show the condition

$$\int_0^T |g_\alpha(T-\tau) A W_\alpha^\Psi(\tau)|_H \, d\tau < +\infty, \quad P - a.s., \quad \alpha > 0, \tag{25}$$

that is, the condition (10) adapted for the fractional Volterra equation (22).

The convolution $W_\alpha^\Psi(t)$ has integrable trajectories, that is, $W_\alpha^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. The closed linear unbounded operator A becomes bounded on $(D(A), |\cdot|_{D(A)})$, see [14, Chapter 5]. Hence, $AW_\alpha^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. Therefore, the function $g_\alpha(T-\tau)AW_\alpha^\Psi(\tau)$ is integrable with respect to τ, what completes the proof. □

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