STOCHASTIC PDEs WITH FUNCTION-VALUED SOLUTIONS*)

ANNA KARCZEWSKA[1] AND JERZY ZABCZYK[2]

[1] Institute of Mathematics
Maria Curie-Skłodowska University
pl. M. Curie-Skłodowskiej 1
20–031 Lublin, Poland
e-mail: akarcz@golem.umcs.lublin.pl

[2] Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00–950 Warszawa, Poland
e-mail: zabczyk@impan.gov.pl

Abstract

The paper provides necessary and sufficient conditions under which stochastic heat and wave equations on $\mathbb{R}^d$ have function-valued solutions. The results extend, to all dimensions d and to all spatially homogeneous perturbations, recent characterizations by Dalang and Frangos [DaFr]. The paper proposes a natural framework for a study of nonlinear stochastic equations. It is based on the harmonic analysis technique and on the stochastic integration theory in functional spaces. Generalizations to the d-dimensional torus and to nonlinear equations are discussed as well.

1 Introduction

The paper is concerned with the stochastic heat and wave equations:

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$$\begin{cases} \frac{\partial u}{\partial t}(t,\theta) = \Delta u(t,\theta) + \frac{\partial W_\Gamma}{\partial t}(t,\theta), \quad t>0, \quad \theta \in \mathbb{R}^d \\ u(0,\theta)=0, \quad \theta \in \mathbb{R}^d \end{cases} \tag{1.1}$$

and

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,\theta) = \Delta u(t,\theta) + \frac{\partial W_\Gamma}{\partial t}(t,\theta), \quad t>0, \quad \theta \in \mathbb{R}^d \\ u(0,\theta)=0, \qquad \frac{\partial u}{\partial t}(0,\theta)=0, \qquad \theta \in \mathbb{R}^d \end{cases} \tag{1.2}$$

where W_Γ is a spatially homogeneous Wiener process with the space correlation Γ. The correlation Γ can be any positive definite distribution. It defines the covariance operator of the Wiener process by the formula $Q\varphi = \Gamma * \varphi$, $\varphi \in S(\mathbb{R}^d)$.

It is well-known, see [Wa], that if $\frac{\partial W_\Gamma}{\partial t}$ is a space-time white noise, or equivalently if $\Gamma = \delta_{\{o\}}$, then the equations (1.1), (1.2) have function-valued solutions if and only if the space dimension $d = 1$. It is therefore of interest to find out in dimensions $d \geqslant 1$ for what space-correlated noise, equations (1.1) and (1.2), have function-valued solutions. This problem has been recently investigated, for stochastic wave equation, by Dalang and Frangos [DaFr], see also Mueller [Mu], when $d = 2$. Let $W_\Gamma(t,\theta), t \geqslant 0, \theta \in \mathbb{R}^2$, be a Wiener process with a space correlation function Γ:

$$\mathbb{E} W_\Gamma(t,\theta) W_\Gamma(s,\eta) = t \wedge s \Gamma(\theta - \eta), \qquad \theta, \eta \in \mathbb{R}^2,$$

where $\Gamma(\theta) = f(|\theta|), \theta \in \mathbb{R}^2$, and f is non-negative function, continuous outside 0. It has been shown in [DaFr] that the stochastic wave equation (1.2) has a function-valued solution if and only if

$$\int_{|\theta|\leqslant 1} f(|\theta|) \ln \frac{1}{|\theta|} d\theta < +\infty. \tag{1.3}$$

The proof in [DaFr] is based on explicit representation of the fundamental solution of the deterministic wave equation in dimension $d = 2$ and can not be extended to higher dimensions.

In the present note we treat the general case of *arbitrary dimension d and of arbitrary spatially homogeneous noise* for both *stochastic heat and wave equations.* Spatially homogeneous noise processes were introduced by Holley and Stroock [HoSt] and Dawson and Salehi [DaSa] in connection with particle systems, see also Nobel [No], Da Prato and Zabczyk [DaPrZa1] and Peszat and Zabczyk [PeZa] for more recent investigations. We consider also equations (1.1) and (1.2) on the d-dimensional torus T^d. It is interesting that for both equations, (1.1) and (1.2) on $\mathbb{R}^d$ and on T^d, the *necessary and sufficient conditions are exactly the same*. Obtained characterizations form a natural framework in which *nonlinear heat and wave equations* can be studied.

Similar results can be formulated for linear parts of Navier-Stokes equations and other equations of fluid dynamics. Techniques developed in the paper apply also to equations (1.1) and (1.2) with Δ replaced by fractional Laplacien $-(-\Delta)^{\alpha}$, $\alpha \in (0,2]$. However, those generalizations are not studied here.

To formulate our main theorems let us recall, see [GeVi], that positive definite, tempered distrubutions Γ are precisely Fourier transforms of tempered measures μ. The measure μ will be called the *spectral measure* of Γ and of the process W_{Γ}.

Theorem 1. *Let Γ be a positive definite, tempered distribution on $\mathbb{R}^d$, with the spectral measure μ. Then the equations (1.1) and (1.2) have function-valued solutions if and only if*

$$\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2}\, \mu(d\lambda) < +\infty. \tag{1.4}$$

Theorem 2. *Assume that Γ is not only a positive definite distribution but also a non-negative measure. The equations (1.1) and (1.2) have function-valued solutions*:

i) for all Γ if $d = 1$;

ii) for exactly those Γ for which $\int_{|\theta|\leqslant 1} \ln|\theta|\Gamma(d\theta) < +\infty$ if $d = 2$;

iii) for exactly those Γ for which $\int_{|\theta|\leqslant 1} \frac{1}{|\theta|^{d-2}}\Gamma(d\theta) < +\infty$ if $d \geqslant 3$.

Note that condition (1.3) is a special case of ii).

Similar theorems hold for stochastic heat and ware equations on the d-dimensional torus, see Theorem 3 and Theorem 4 in §5.

The paper is organized as follows. Preliminaries and formulation of the problem will be given in section 2. Section 3 contains proofs of the results for the case of $\mathbb{R}^d$. Applications are discussed in section 4. Extensions to d-dimensional torus are contained in section 5. We finish the paper with two conjectures in section 6.

2 Preliminaries

2.1 Heat and wave semigroups

Let $S_c(\mathbb{R}^d)$ denote the space of all infinitely differentiable functions ψ on $\mathbb{R}^d$ taking complex values, for which the seminorms

$$\| \psi \|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^d} |x^{\alpha} D^{\beta}\psi(x)|$$

are finite. The adjoint space $S_c'(\mathbb{R}^d)$ is then the space of tempered distributions. By $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ we denote the spaces of real functions from $S_c(\mathbb{R}^d)$ and the space of real functionals on $S(\mathbb{R}^d)$.

For $\psi \in S_c(\mathbb{R}^d)$ define $\psi_{(s)}(x) = \overline{\psi(-x)}, x \in \mathbb{R}^d$. By $S_{(s)}(\mathbb{R}^d)$ and $S'_{(s)}(\mathbb{R}^d)$ denote the spaces of $\psi \in S_c(\mathbb{R}^d)$ such that $\psi(x) = \psi_{(s)}(x)$ and the space of all $\xi \in S_c'(\mathbb{R}^d)$ such that $(\xi, \psi) = (\xi, \psi_{(s)})$ for all $\psi \in S(\mathbb{R}^d)$.

If $\mathcal{F}$ is the Fourier transform on $S_c(\mathbb{R}^d)$:

$$\mathcal{F}(\psi)(\lambda) = \int_{\mathbb{R}^d} e^{t\langle x,\lambda\rangle}\psi(x)dx, \quad \lambda \in \mathbb{R}^d, \quad \psi \in S_c(\mathbb{R}^d),$$

then its inverse $\mathcal{F}^{-1}$ is given by the formula

$$\mathcal{F}^{-1}\psi(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-t\langle x,\lambda\rangle}\psi(\lambda)d\lambda, \quad \lambda \in \mathbb{R}^d, \quad \psi \in S_c(\mathbb{R}^d).$$

We use the same notation for the Fourier transforms acting on $S_c'(\mathbb{R}^d)$.
Note that the operators $\mathcal{F}$ and $\mathcal{F}^{-1}$ transform $S'(\mathbb{R}^d)$ onto $S'_{(s)}(\mathbb{R}^d)$ and $S'_{(s)}(\mathbb{R}^d)$ onto $S'(\mathbb{R}^d)$, respectively. The Fourier transforms of $\varphi \in S_c(\mathbb{R}^d)$ and $\xi \in S_c'(\mathbb{R}^d)$ will be denoted by $\hat{\varphi}$ and $\hat{\xi}$.

Consider first the heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \quad t \geqslant 0 \quad u(0) = \xi \tag{2.1}$$

where $\xi \in S_c'(\mathbb{R}^d)$. If $\hat{u}$ denotes the Fourier transform of u then

$$\frac{\partial \hat{u}}{\partial t} = -|\lambda|^2\hat{u} \quad \text{and} \quad \hat{u}(0) = \hat{\xi},$$

and therefore

$$\hat{u}(t) = e^{-|\lambda|^2 t}\hat{\xi}.$$

Consequently, for arbitrary $\xi \in S_c'(\mathbb{R}^d)$, equation (2.1) has a unique solution in $S_c'(\mathbb{R}^d)$ and the solution is given by the formula

$$u(t,x) = p(t) * \xi(x) = (\xi, p(t, x - \cdot)) \tag{2.2}$$

where $\hat{p}(t)(\lambda) = e^{-t|\lambda|^2}, \quad p(t,x) = \frac{1}{\sqrt{(4\pi t)^d}}e^{\frac{-|x|^2}{4t}}, \quad t > 0, \quad \lambda, x \in \mathbb{R}^d.$

The family

$$S(t)\xi = p(t) * \xi, \quad t \geqslant 0, \quad \xi \in S_c'(\mathbb{R}^d) \tag{2.3}$$

forms a semigroup of operators, continuous in the topology of $S_c'(\mathbb{R}^d)$. The formula (2.2) shows that the semigroup $S(t)$, $t \geqslant 0$ has a smoothing property: for all $\xi \in S_c'(\mathbb{R}^d)$, $S(t)\xi$ is represented by C^∞ function.

Similarly, for the wave equation,

$$\tfrac{\partial u}{\partial t} = v, \qquad u(0) = \xi$$

$$\tfrac{\partial v}{\partial t} = \Delta u, \qquad v(0) = \zeta,$$

one gets, passing again to the Fourier transforms $\hat{u}$ and $\hat{v}$, that:

$$\frac{\partial \hat{u}}{\partial t} = \hat{v}, \quad \frac{\partial \hat{v}}{\partial t} = -|\lambda|^2 \hat{u}.$$

By direct computation we have

$$\frac{d}{dt}\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} \cos\,(|\lambda| t)\,, & \frac{\sin(|\lambda| t)}{|\lambda|} \\ -|\lambda| \sin(|\lambda| t), & \cos\,(|\lambda| t) \end{pmatrix} \begin{pmatrix} \hat{\xi} \\ \hat{\zeta} \end{pmatrix}.$$

Therefore,

$$\hat{u}(t) = [\cos(|\lambda| t)]\hat{u}(0) + \left[\frac{\sin(|\lambda| t)}{|\lambda|}\right]\hat{v}(0), \tag{2.4}$$

$$\hat{v}(t) = -[|\lambda| \sin(|\lambda| t)]\hat{u}(0) + [\cos(|\lambda| t)]\hat{v}(0). \tag{2.5}$$

Note that for each $t \in \mathbb{R}^1$ functions $\cos(|\lambda| t)$, $\frac{\sin(|\lambda| t)}{|\lambda|}$ and $|\lambda| \sin(|\lambda| t)$, $\lambda \in \mathbb{R}^d$, are smooth and polynomially bounded together with all their partial derivatives. Therefore the formulae (2.4), (2.5) define distributions belonging to $S_c'(\mathbb{R}^d)$.

Let $p_{1,1}(t)$, $p_{1,2}(t)$, $p_{2,1}(t)$, $p_{2,2}(t)$ $t \in \mathbb{R}^1$, be elements from $S'(\mathbb{R}^d)$ such that,

$$\cos(|\lambda| t) = \mathcal{F}\left(p_{1,1}(t)\right)(\lambda), \quad \frac{\sin(|\lambda| t)}{|\lambda|} = \mathcal{F}\left(p_{1,2}(t)\right)(\lambda)$$

$$-|\lambda| \sin(|\lambda| t) = \mathcal{F}\left(p_{2,1}(t)\right)(\lambda), \quad \cos(|\lambda| t) = \mathcal{F}\left(p_{2,2}(t)\right)(\lambda),\ \lambda \in \mathbb{R}^d.$$

Then

$$u(t) = p_{1,1}(t) * u(0) + p_{1,2}(t) * v(0),$$

$$v(t) = p_{2,1}(t) * u(0) + p_{2,2}(t) * v(0), \quad t \in \mathbb{R}.$$

As for the heat equation, explicit formulae for the distributions $p_{i,j}(t)$, $i,j=1,2$ are known, see [Mi, pp. 280–282]. In particular, they have bounded supports.

We shall use the following notation

$$R(t)\xi = p_{1,2}(t) * \xi, \quad t \geqslant 0, \xi \in S'_c(\mathbb{R}^d). \tag{2.6}$$

2.2 Spatially homogeneous Wiener process

Let Γ be a positive definite, tempered distribution. By W_Γ we denote an $S'(\mathbb{R}^d)$-valued Wiener process defined on a probability space $(\Omega, F, \mathbb{P})$ such that

$$\mathbb{E}(W(t),\varphi)(W(s),\psi) = t \wedge s(\Gamma, \varphi * \psi_{(s)}),$$

where $\psi_{(s)}(x) = \overline{\psi(-x)}$, $x \in \mathbb{R}^d$, see [PeZa]. It is well-known that this way one can describe all space homogeneous $S'(\mathbb{R}^d)$-valued Wiener processes, see e.g. [PeZa].

The crucial role for stochastic integration with respect to W_Γ is played by the Hilbert space $S'_\Gamma \subset S'(\mathbb{R}^d)$ consisting of all distributions $\xi \in S'(\mathbb{R}^d)$ for which there exists a constant C such that,

$$|(\xi,\psi)| \leqslant C\sqrt{(\Gamma, \psi * \psi_{(s)})}, \quad \psi \in S.$$

The norm in S'_Γ is given by the formula:

$$|\xi|_{S'_\Gamma} = \sup_{\psi \in S} \frac{|(\xi,\psi)|}{\sqrt{(\Gamma, \psi * \psi_{(s)})}}.$$

The space S'_Γ is called the *kernel* of W_Γ, see [PeZa].

Let H be a Hilbert space and let $L_{HS}(S'_\Gamma, H)$ be the space of Hilbert-Schmidt operators from S'_Γ into H. Assume that Ψ is a predictable $L_{HS}(S'_\Gamma, H)$-valued process such that

$$\mathbb{E}\left(\int_0^t \parallel \Psi(s) \parallel^2_{L_{HS}(S'_\Gamma,H)} ds\right) < +\infty \quad \text{for all } t \geqslant 0.$$

Then the stochastic integral

$$\int_0^t \Psi(s)dW_\Gamma(s), \quad t \geqslant 0$$

can be defined in a standard way, see [Itô], [DaPrZa], [PeZa]. It is an H–valued martingale for which

$$\mathbb{E}\left(\int_0^t \Psi(s)dW_\Gamma(s)\right) = 0, \quad t \geqslant 0$$

and

$$\mathbb{E}\left|\int_0^t \Psi(s)dW_\Gamma(s)\right|_H^2 = \mathbb{E}\left(\int_0^t \| \Psi(s) \|^2_{L_{HS}(S'_\Gamma,H)} ds\right), \quad t \geqslant 0.$$

We will need a characterization of the space S'_Γ from [PeZa, Proposition 1.2]. In the proposition below $L^2_{(s)}(\mathbb{R}^d,\mu)$ denotes the subspace of $L^2(\mathbb{R}^d,\mu;\mathbb{C})$ consisting of all functions u such that $u_{(s)}=u$, see §2.1.

Proposition 1. *A distribution ξ belongs to S'_Γ if and only if $\xi=\widehat{u\mu}$ for some $u\in L^2_{(s)}(\mathbb{R}^d,\mu)$. Moreover, if $\xi=\widehat{u\mu}$ and $\eta=\widehat{v\mu}$, then*

$$\langle \xi,\eta\rangle_{S'_\Gamma} = \langle u,v\rangle_{L^2_{(s)}(\mathbb{R}^d,\mu)}.$$

2.3 Questions

By a *solution* X to the *stochastic heat equation* we understand the process

$$X(t)=\int_0^t S(t-s)dW_\Gamma(ds), \quad t>0, \tag{2.7}$$

where $S(\cdot)$ is given by (2.3). Similarly, *solution* Y to the *stochastic wave equation* is of the form

$$Y(t)=\int_0^t R(t-s)dW_\Gamma(ds), \quad t\geqslant 0, \tag{2.8}$$

with $R(\cdot)$ defined by (2.6). It is not difficult to show that the processes $X(t)$, $t\geqslant 0$, and $Y(t)$, $t\geq 0$, are weak solutions of the corresponding equations and take values in $S'(\mathbb{R}^d)$, see [Itô].

Let us recall that a family $Z(x)$, $x\in\mathbb{R}^d$, of real random variables is called a *stationary, Gaussian, random field* if and only if, Z is a measurable transformation from $\mathbb{R}^d\times\Omega$ into $\mathbb{R}$ and for arbitrary $h,x_1,\ldots,x_m\in\mathbb{R}^d$, random vectors $(Z(x_1+h),\ldots,Z(x_m+h))$ are Gaussian with the law independent of h.

The main questions considered in the paper can be stated as follows.

Question 1. Under what conditions on Γ, for each $t\geqslant 0$, $X(t)$ is a stationary, Gaussian random field?

Question 2. Under what conditions on Γ, for each $t\geqslant 0$, $Y(t)$ is a stationary, Gaussian random field?

Note that if Z is a stationary, Gaussian random field then, for all positive, integrable functions $\rho(x)$, $x\in\mathbb{R}^d$

$$\mathbb{E}\left(\int_{\mathbb{R}^d} Z^2(x)\rho(x)dx\right) = \int_{\mathbb{R}^d} (\mathbb{E}Z^2(x))\rho(x)dx = \left(\int_{\mathbb{R}^d} \rho(x)dx\right) \mathbb{E}(Z^2(o)) < +\infty.$$

Consequently, $\mathbb{P}(Z \in L^2_\rho(\mathbb{R}^d)) = 1$, where $L^2_\rho(\mathbb{R}^d) = L^2(\mathbb{R}^d, \rho(x)dx)$ and the questions can be reformulated as follows.

Question 1. Under what conditions on Γ the process X takes values in $L^2_\rho(\mathbb{R}^d)$ for some (any) positive integrable weight ρ ?

Question 2. Under what conditions on Γ the process Y takes values in $L^2_\rho(\mathbb{R}^d)$ for some (any) positive integrable weight ρ ?

Answers to these questions have been formulated in the Introduction as Theorem 1 and Theorem 2. The case of d-dimensional forms T^d is treated in §5.

3 Proofs of Theorem 1 and Theorem 2

3.1 Proof of Theorem 1.

(a) Stochastic heat equation

Let us recall that we denote by S'_Γ the kernel of the Wiener process W_Γ and that

$$S(t)\xi = p(t) * \xi, \quad t \geqslant 0.$$

It follows from §2.2 that the stochastic integral

$$\int_0^t S(t-s)dW_\Gamma(s), \quad t > 0$$

takes values in $L^2_\rho(\mathbb{R}^d)$ if and only if

$$\int_0^t \| S(\sigma) \|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} d\sigma < +\infty.$$

Let $\{u_k\}$ be an orthonormal basis in $L^2_{(s)}(\mathbb{R}^d, \mu)$. Then by Proposition 1.2 of [PeZa], $e_k = \widehat{u_k\mu}$, $k \in \mathbb{N}$, is an orthonormal basis in S'_Γ.
Thus we have

$$\| S(\sigma) \|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} = \sum_{k=1}^\infty |S(\sigma)\widehat{u_k\mu}|^2_{L^2_\rho} = \sum_{k=1}^\infty \int_{\mathbb{R}^d} |p(\sigma) * \widehat{u_k\mu}(x)|^2 \rho(x)dx, \quad \sigma > 0.$$

However, $p(\sigma) \in S(\mathbb{R}^d)$ and therefore

$$p(\sigma) * \widehat{u_k\mu}(x) = (p(\sigma, x - \cdot), \widehat{u_k\mu}) = (u_k\mu, \hat{p}(\sigma, x - \cdot))\,.$$

The last identity follows from the definition of the Fourier transform of the distribution $u_k\mu$. However,

$$\hat{p}(\sigma, x - \cdot)(\lambda) = e^{i\langle x,\lambda\rangle} e^{-\sigma|\lambda|^2}$$

and therefore

$$(u_k\mu, \hat{p}(\sigma, x - \cdot)) = (u_k\mu, e^{i\langle x,\cdot\rangle} e^{-\sigma|\cdot|^2}).$$

Consequently

$$\| S(\sigma) \|^2_{L_{HS}(S'_\Gamma L^2_\rho)} = \sum_k \int_{\mathbb{R}^d} \left|(u_k\mu, e^{i\langle x,\cdot\rangle} e^{-\sigma|\cdot|^2})\right|^2 \rho(x)dx$$

$$= \sum_k \int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} u_k(\lambda) e^{i\langle x,\lambda\rangle} e^{-\sigma|\lambda|^2} \mu(d\lambda)\right|^2 \rho(x)dx$$

$$= \int_{\mathbb{R}^d} \left[\sum_k \left|\langle u_k, e^{-i\langle x,\cdot\rangle} e^{-\sigma|\cdot|^2}\rangle_{L^2_{(s)}(\mathbb{R}^d,\mu)}\right|^2\right] \rho(x)dx.$$

By the Parseval identity in $L^2_{(s)}(\mathbb{R}^d, \mu)$,

$$\sum_k \left|\langle u_k, e^{-i\langle x,\cdot\rangle} e^{-\sigma|\cdot|^2}\rangle_{L^2_{(s)}(\mathbb{R}^d,\mu)}\right|^2 = \int_{\mathbb{R}^d} \left|e^{-i\langle x,\lambda\rangle} e^{-\sigma|\lambda|^2}\right|^2 \mu(d\lambda) = \int_{\mathbb{R}^d} e^{-2\sigma|\lambda|^2} \mu(d\lambda).$$

Finally,

$$\int_0^t \| S(\sigma) \|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} d\sigma = \left[\int_{\mathbb{R}^d} \rho(x)dx\right] \int_0^t \int_{\mathbb{R}^d} e^{2\sigma|\lambda|^2} \mu(d\lambda)d\sigma =$$

$$= \left[\int_{\mathbb{R}^d} \rho(x)dx\right] \int_{\mathbb{R}^d} \frac{1 - e^{-2t|\lambda|^2}}{|\lambda|^2} \mu(d\lambda).$$

Therefore

$$\int_0^t \| S(\sigma) \|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} d\sigma < +\infty$$

if and only if

$$\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2}\mu(d\lambda) < +\infty.$$

(b) Stochastic wave equation

Let us recall that $R(\sigma)\xi = p_{1,2}(\sigma) * \xi$, $\sigma \geqslant 0$, $\xi \in S'_c(\mathbb{R}^d)$, see (2.6). The process $Y(t)$, $t \geqslant 0$, is well defined as an $L^2_\rho(\mathbb{R}^d)$-valued process if and only if

$$\int_0^t \| R(\sigma) \|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} d\sigma < +\infty.$$

But

$$\| R(\sigma) \|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} = \sum_k \int_{\mathbb{R}^d} |p_{1,2}(\sigma) * \widehat{u_k\mu}(x)|^2 \rho(x) dx.$$

However

$$p_{1,2}(\sigma) * \widehat{u_k\mu}(x) = (p_{1,2}(\sigma, x-\cdot), \widehat{u_k\mu}) = (\hat{p}_{1,2}(\sigma)(x-\cdot), u_k\mu). \tag{3.1}$$

To justify the identity (3.1) we need the following lemma, see [GeSh].

Lemma 1. *Let ξ and η be distributions with bounded supports. Then the convolution $\xi * \hat{\eta}$ exists and is a function of class C^∞. Moreover*

$$\xi * \hat{\eta}(x) = (\hat{\xi}(x-\cdot), \eta), \quad x \in \mathbb{R}^d.$$

Note that the distribution $p_{1,2}$ has bounded support and one can assume also that functions u_k, $k \in \mathbb{N}$, have bounded supports as well.
But

$$\hat{p}_{1,2}(\sigma)(x-\cdot)(\lambda) = e^{i\langle x,\lambda\rangle} \frac{\sin(|\lambda|\sigma)}{|\lambda|}.$$

Therefore

$$\| R(\sigma) \|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} = \sum_k \int_{\mathbb{R}^d} \left| \left(u_k\mu, e^{i\langle x,\cdot\rangle} \frac{\sin(|\cdot|\sigma)}{|\cdot|} \right) \right|^2 \rho(x) dx =$$

$$= \sum_k \int_{\mathbb{R}^d} \left| \langle u_k, e^{i\langle x,\cdot\rangle} \frac{\sin(|\cdot|\sigma)}{|\cdot|} \rangle_{L^2_{(s)}(\mathbb{R}^d,\mu)} \right|^2 \rho(x) dx.$$

Again, by the Parseval identity,

$$\| R(\sigma) \|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} = \left[\int_{\mathbb{R}^d} \rho(x) dx \right] \int_{\mathbb{R}^d} \frac{(\sin(|\lambda|\sigma))^2}{|\lambda|^2} \mu(d\lambda).$$

Consequently,

$$\int_0^t \| R(\sigma) \|^2_{L_{HS}(S'_\Gamma, L^2_\rho)} d\sigma = \left[\int_{\mathbb{R}^d} \rho(x) dx \right] \int_{\mathbb{R}^d} \left[\int_0^t \frac{(\sin(|\lambda|\sigma))^2}{|\lambda|^2} d\sigma \right] \mu(d\lambda).$$

By an elementary argument one shows now that the integral is finite, for all $t > 0$, if and only if

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^2} \mu(d\lambda) < +\infty.$$

This completes the proof of Theorem 1.

■

3.2 Proof of Theorem 2.

Let

$$G_d(x) = \int_0^{+\infty} e^{-t} p(t, x) dt, \quad x \in \mathbb{R}^d$$

where

$$p(t, x) = \frac{1}{\sqrt{(4\pi t)^d}} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^d.$$

Thus G_d is the resolvent kernel of the d-dimensional Wiener process. It is easy to see that

$$G_d(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle x, \lambda \rangle} \frac{1}{1 + |\lambda|^2} d\lambda, \quad x \in \mathbb{R}^d.$$

The following properties of G_d are well-known, see [La], [GlJa], [GeSh]:

Proposition 2. *One has that:*

$$G_1(x) = \frac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}^1; \qquad G_3(x) = \frac{1}{4\pi|x|} e^{-|x|}, \quad x \in \mathbb{R}^3$$

and, in general, for $d \geqslant 2$,

$$G_d(x) = (2\pi)^{-\frac{d}{2}} \frac{1}{|x|^{\frac{d-2}{2}}} K_{\frac{d-2}{2}}(|x|),$$

where K_γ, $\gamma \geqslant 0$, denotes the modified Bessel function of the third order.

We will need also a characterization of the behaviour of G_d near 0 and near ∞, see [GlJa, Proposition 7.2.1].

Proposition 3. *The function G_d has the following properties:*
(a) for $d \geqslant 1$, for $|x|$ bounded away from a neighbourhood of zero and for a constant $c > 0$

$$G_d(x) \leqslant \frac{c}{|x|^{\frac{d-1}{2}}} e^{-|x|};$$

(b) for $d \geqslant 3$ and for a constant $c > 0$, in a neighbourhood of zero

$$G_d(x) \sim \frac{c}{|x|^{d-2}};$$

(c) for $d = 2$ and for a constant $c > 0$, in a neighbourhood of zero

$$G_2(x) \sim \; - c \ln |x|.$$

We will need also the following lemma:

Lemma 2. *Assume that the distribution Γ is not only positive definite but it is also a non-negative measure. Then*

$$(\Gamma, G_d) = (2\pi)^d \int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^2} \mu(d\lambda).$$

Proof of Lemma 2. Since $\mu = \mathcal{F}^{-1}(\Gamma)$ and $e^{-t|\cdot|^2} \frac{1}{1+|\cdot|^2} \in S(\mathbb{R}^d)$, by the definition of the Fourier transform of a distribution,

$$\int_{\mathbb{R}^d} \frac{e^{-t|\lambda|^2}}{1 + |\lambda|^2} \mu(d\lambda) = \left(\frac{e^{-t|\cdot|^2}}{1 + |\cdot|^2}, \mu \right) = \left(\frac{e^{-t|\cdot|^2}}{1 + |\cdot|^2}, \mathcal{F}^{-1}(\Gamma) \right) =$$

$$= \left(\mathcal{F}^{-1} \left(\frac{e^{-t|\cdot|^2}}{1 + |\cdot|^2} \right), \Gamma \right) = \frac{1}{(2\pi)^d} \left(p(t) * G_d, \Gamma \right).$$

Therefore

$$\lim_{t \downarrow 0} \frac{1}{(2\pi)^d} (p(t) * G_d, \Gamma) = \int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^2} \mu(d\lambda).$$

Moreover

$$p(s) * G_d = \int_0^{+\infty} e^{-t} p(t) * p(s) dt =$$

$$= e^s \int_0^{+\infty} e^{-(t+s)} p(t+s) dt = e^s \int_s^{+\infty} e^{-\sigma} p(\sigma) d\sigma.$$

So

$$e^{-s} p(s) * G_d = \int_s^{+\infty} e^{-\sigma} p(\sigma) d\sigma$$

and then

$$e^{-s} p(s) * G_d \uparrow G_d \quad \text{as} \quad s \downarrow 0.$$

Hence, if Γ is a non-negative distribution on $\mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2} \mu(d\lambda) = \lim_{t \downarrow 0} e^{-t} \frac{1}{(2\pi)^d} (p(t) * G_d, \Gamma) = \frac{1}{(2\pi)^d} (G_d, \Gamma).$$

This completes the proof of Lemma 2.

■

We pass now to the proof of the theorem. It is well-known that a non-negative measure Γ belongs to $S'(\mathbb{R}^d)$ if and only if for some $r > 0$,

$$\int_{\mathbb{R}^d} \frac{1}{1+|x|^r} \Gamma(dx) < +\infty. \tag{3.2}$$

Moreover, for arbitrary $d \geqslant 1$,

$$\int_{\mathbb{R}^d} G_d(x) \Gamma(dx) = \int_{|x| \leqslant 1} G_d(x) \Gamma(dx) + \int_{|x|>1} G_d(x) \Gamma(dx).$$

But, by Proposition 3 (a),

$$\int_{|x|>1} G_d(x) \Gamma(dx) \leqslant c \int_{|x|>1} e^{-|x|} \Gamma(dx)$$

and from (3.2)

$$\int_{|x|>1} G_d(x) \Gamma(dx) < +\infty.$$

Since the function G_1 is continuous,

$$\int_{|x|\leqslant 1} G_1(x)\Gamma(dx) < +\infty$$

and the theorem is true for $d=1$.

If $d=2$ then $\int_{\mathbb{R}^d} G_2(x)\Gamma(dx) < +\infty$ if and only if $\int_{|x|\leqslant 1} G_2(x)\Gamma(dx) < +\infty$. But $G_2(x) \sim c \ln \frac{1}{|x|}$ for some $c > 0$ in the neighbourhood of 0, so

$$G_2(x)/c \ln \frac{1}{|x|} \to 1 \text{ as } |x| \to 0.$$

Therefore, for some $c_1 > 0$, $c_2 > 0$:

$$c_2 \ln \frac{1}{|x|} \leqslant G_2(x) \leqslant c_1 \ln \frac{1}{|x|}, \text{ for } |x| \leqslant 1.$$

Consequently,

$$\int_{\mathbb{R}^d} G_2(x)\Gamma(dx) < +\infty \text{ if and only if } \int_{|x|\leqslant 1} \ln \frac{1}{|x|}\Gamma(dx) < +\infty.$$

If $d \geqslant 3$, in the same way,

$$\int_{\mathbb{R}^d} G_d(x)\Gamma(dx) < +\infty \text{ if and only if } \int_{\mathbb{R}^d} \frac{1}{|x|^{d-2}}\Gamma(dx) < +\infty.$$

This completes the proof of Theorem 2.

■

4 Applications

We illustrate the main results by several examples. We start with the case of bounded functions Γ.

Proposition 4. *If the positive definite distribution Γ is a bounded function then the equations (1.1) and (1.2) have function-valued solutions in any dimension d.*

Proof: If the positive definite distribution Γ is a bounded function then Γ is a continuous function and the corresponding spectral measure μ is finite. Since the function $\frac{1}{1+|\lambda|^2}$, $\lambda \in \mathbb{R}^d$, is bounded therefore

$$\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2} \mu(d\lambda) < +\infty$$

and by Theorem 1 the result follows.

■

Stochastic evolution equations with noise of such type have been introduced by Dawson and Salahi [DaSa] with an extra requirement that μ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$. In the case of $d=2$ they have appeared in the recent paper by Mueller [Mu].

Example 1. It is well-known that functions $\Gamma(x) = e^{-|x|^\alpha}$, $x \in \mathbb{R}^d$, for $\alpha \in (0, 2]$ are positive definite. In fact, they are Fourier transforms of the so called symmetric stable distributions, see [La] or [Fe]. Consequently with such covariances Γ the equations (1.1) and (1.2) have function-valued solutions.

We consider now some examples of unbounded covariances Γ.

Proposition 5. *For arbitrary $\alpha \in (0, d)$ the function $\Gamma_\alpha(x) = \frac{1}{|x|^\alpha}$, $x \in \mathbb{R}^d$ is a positive definite distribution. Equations (1.1) and (1.2) with the covariance Γ_α have function-valued solutions if and only if $\alpha \in (0, 2 \wedge d)$.*

Proof. It is well-known, see [Mi], [GeSh] or [La], that Γ_α is the Fourier transform of the function $c_1 \frac{1}{|\lambda|^{d-\alpha}}$, $\lambda \in \mathbb{R}^d$, where c_1 is a positive constant. The condition (1.4) is equivalent to

$$I := \int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2} \frac{1}{|\lambda|^{d-\alpha}} d\lambda < +\infty.$$

By standard calculation

$$I = c_2 \int_0^{+\infty} \frac{1}{(1+r^2)} \frac{1}{r^{d-\alpha}} r^{d-1} dr,$$

where c_2 is a constant. One obtains, that $I < +\infty$ if and only if

$$\int_0^1 \frac{1}{r^{1-\alpha}} dr < +\infty \quad \text{and} \quad \int_1^\infty \frac{1}{r^{3-\alpha}} dr < +\infty,$$

or equivalently, $\alpha > 0$ and $\alpha < 2$. Since $\alpha \in (0, d)$, the result follows.

■

Remark: Note that Proposition 5 contains, as a special case, an application from the paper [DaFr, see Examples].

We pass now to examples for which Γ are genuine distributions.

Example 2. If $\frac{\partial W_\Gamma}{\partial t}$ is the space-time white noise then $\Gamma = \delta_{\{o\}}$ and the corresponding spectral measure μ has a constant density, say $c > 0$. Since

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\lambda|^2} d\lambda < +\infty$$

if and only if $d = 1$, the equations (1.1) and (1.2), perturbed by such noise, have function-valued solutions iff $d = 1$.

Example 3. Walsh [Wa], in his study of particle systems, arrived at the following equation for fluctuations:

$$\frac{\partial u}{\partial t}(t, \xi) = \frac{\partial^2 u}{\partial \xi^2}(t, \xi) + \frac{\partial}{\partial t}\left[\frac{\partial}{\partial \xi} W_{\delta_{\{o\}}}(t, \xi)\right] \tag{4.1}$$

$$u(0, \xi) = 0, \quad t > 0, \quad \xi \in \mathbb{R}^1.$$

It is easy to calculate that the covariance function corresponding to $\frac{\partial}{\partial \xi} W_{\delta_{\{o\}}}(t, \xi)$, $t \geqslant 0$, is $\Gamma = -\delta''_{\{0\}}$ and the appropriate spectral measure μ has the following density $d_\mu(\lambda) = \lambda^2 d\lambda$.
Since

$$\int_{-\infty}^{+\infty} \frac{1}{1 + \lambda^2} \lambda^2 d\lambda = +\infty$$

the equation (4.1) does not have a function-valued solution. This fact has already been noticed by Walsh [Wa].

5 Equations on d-dimensional torus

Many of the previous considerations can be extended from $\mathbb{R}^d$ to stochastic equations on more general groups. As an illustration we discuss here the case of d-dimensional torus T^d, for more details we refer to [KaZa]. The d-dimensional torus T^d can be identified with the Cartesian product, $(-\pi, \pi]^d$, regarded as a group with the addition modulo 2π (coordinate-wise). We assume that W_Γ is a $D'(T^d)$-valued Wiener process spatially homogeneous with the space correlation Γ. Distribution Γ can be uniquelly expanded into its Fourier series

$$\Gamma(\theta) = \sum_{n\in\mathbb{Z}^d} e^{i\langle n,\theta\rangle}\gamma_n$$

with the non-negative coefficients such that $\gamma_n = \gamma_{-n}$ and $\sum\limits_{n\in\mathbb{Z}^d} \frac{\gamma_n}{1+|n|^{+\infty}} < r$ for some $r > 0$.

Denote $\mathbb{Z}_s^1 = \mathbb{N}$ and, by induction, $\mathbb{Z}_s^{d+1} = \left(\mathbb{Z}_s^1 \times \mathbb{Z}^d\right) \cup \left\{(0,n);\ n \in \mathbb{Z}_s^d\right\}$. Then $\mathbb{Z}^d = \mathbb{Z}_s^d \cup \left(-\mathbb{Z}_s^d\right) \cup \{0\}$.

The corresponding spatially homogeneous Wiener process $W(t)$, $t \geqslant 0$ can be represented in the form:

$$W(t,\theta) = \sqrt{\gamma_0}\beta_0(t) \quad + \sum_{n\in\mathbb{Z}_s^d} \sqrt{2\gamma_n}\left((\cos\langle n,\theta\rangle)\beta_n^1(t) + (\sin\langle n,\theta\rangle)\beta_n^2(t)\right), \qquad \theta \in T^d, \quad t \geqslant 0 \tag{5.1}$$

where $\beta_0, \beta_n^1, \beta_n^2$, $n \in \mathbb{Z}_s^d$ are independent, real Brownian motions and the convergence is in the sense of $D'(T_d)$.

Denote $H = H^0 = L^2(T^d)$, $H^\alpha = H^\alpha(T^d)$ and $H^{-\alpha} = H^{-\alpha}(T^d)$, $\alpha \in \mathbb{R}_+$, the real Sobolev spaces of order α and $-\alpha$, respectively. The norms are expressed in terms of the Fourier coefficients, see [Ad]

$$\|\ \xi\ \|_{H_{-\alpha}} = \left(\sum_{n\in\mathbb{Z}^d}(1+|n|^2)^\alpha|\xi_n|^2\right)^{\frac{1}{2}} = \left(|\xi_0|^2 + 2\sum_{n\in\mathbb{Z}_s^d}(1+|n|^2)^\alpha\left((\xi_n^1)^2 + (\xi_n^2)^2\right)\right)^{\frac{1}{2}},$$

and

$$\|\ \xi\ \|_{H^{-\alpha}} = \left(\sum_{n\in\mathbb{Z}^d}(1+|n|^2)^{-\alpha}|\xi_n|^2\right)^{\frac{1}{2}} = \left(|\xi_0|^2 + 2\sum_{n\in\mathbb{Z}_s^d}(1+|n|^2)^{-\alpha}\left((\xi_n^1)^2 + (\xi_n^2)^2\right)\right)^{\frac{1}{2}},$$

where $\xi_n = \xi_n^1 + i\xi_n^2$, $\xi_n = \overline{\xi}_{-n}$, $n \in \mathbb{Z}^d$.

We have the following result

Theorem 3. *Equations (1.1) and (1.2) on the torus T^d have $H^{\alpha+1}(T^d)$-valued solution if and only if the Fourier coefficients (γ_n) of the kernel Γ satisfy:*

$$\sum_{n\in\mathbb{Z}^d} \frac{\gamma_n}{(1+|n|^2)^\alpha} < +\infty. \tag{5.2}$$

Remark: Recently, the problem of existence of solution to stochastic wave equation in $S'(\mathbb{R}^d)$ has been recently considered by Gaveau [Ga].

As in the case of $\mathbb{R}^d$, condition (5.1) can be written in a more explicit way.

Theorem 4. *Assume that Γ is not only a positive definite distribution but is also a non-negative measure. Then equations (1.1) and (1.2) have function-valued solutions*:

i) for all Γ if $d=1$;

ii) for exactly those Γ for which $\int_{|\theta|\leqslant 1} \ln|\theta|\Gamma(d\theta) < +\infty$ if $d=2$;

iii) for exactly those Γ for which $\int_{|\theta|\leqslant 1} \frac{1}{|\theta|^{d-2}}\Gamma(d\theta) < +\infty$ if $d \geqslant 3$.

The proofs of both theorems are similar to those for $\mathbb{R}^d$. For details we refer to our preprint [KaZa].

In fact the proof of Theorem 3 can be done in a different way by taking into account the expansion (5.1) of the Wiener process W, with respect to the basis 1, $\cos\langle n,\theta\rangle$, $\sin\langle n,\theta\rangle$, $n\in\mathbb{Z}^d_s$, $\theta\in T^d$. Equations (1.1) and (1.2) can be solved coordinatwise with the following explicit formulae for the solutions:

$$\begin{aligned} X(t,\theta) &= \sqrt{\gamma_0}\beta_0(t) + \sum_{n\in\mathbb{Z}^d_s} \sqrt{2\gamma_n}\left[\cos\langle n,\theta\rangle \int_0^t e^{-|n|^2(t-s)} d\beta^1_n(s)\right. \\ &+ \left.\sin\langle n,\theta\rangle \int_0^t e^{-|n|^2(t-s)} d\beta^2_n(s)\right], \end{aligned} \tag{5.3}$$

$$\begin{aligned} Y(t,\theta) &= \sqrt{\gamma_0}\beta_0(s)ds + \sum_{n\in\mathbb{Z}^d_s} \sqrt{2\gamma_n}\left[\cos\langle n,\theta\rangle \int_0^t \frac{\sin(|n|(t-s)}{|n|} d\beta^1_n(s)\right. \\ &+ \left.\sin\langle n,\theta\rangle \int_0^t \frac{\sin|n|(t-s)}{|n|} d\beta^2_n(s)\right]. \end{aligned} \tag{5.4}$$

Therefore

$$\begin{aligned}
\mathbb{E}|X(t)|_H^2 &= (2\pi)^d \left[\gamma_0 t + \sum_{n\in\mathbb{Z}_s^d} 2\gamma_n \int_0^t e^{-2|n|^2 s} ds\right], \\
\mathbb{E}|Y(t)|_H^2 &= (2\pi)^d \left[\gamma_0 \frac{t^3}{3} + \sum_{n\in\mathbb{Z}_s^d} \frac{2\gamma_n}{|n|^2} \int_0^t \sin^2(|n|s) ds\right], \ t \geqslant 0.
\end{aligned}$$

Since, for arbitrary $t > 0$,

$$|n|^2 \int_0^t e^{-2|n|^2 s} ds \to \frac{1}{2}, \text{ as } |n| \to +\infty,$$

$$\int_0^t \sin^2(|n|s) ds \to \int_0^t \sin^2 \sigma d\sigma, \text{ as } |n| \to +\infty,$$

therefore $\mathbb{E}|X(t)|_H^2 < +\infty$, $\mathbb{E}|Y(t)|_H^2 < +\infty$ if and only if $\sum_{n\in\mathbb{Z}_{(s)}^d} \frac{\gamma_n}{|n|^2} < +\infty$, as required, ($\alpha = 0$).

Expansions (5.1), (5.2) lead also to more refined results.

Theorem 5. *Assume that*

$$\sum_{n\in\mathbb{Z}^d} \frac{\gamma_n}{1+|n|^\alpha} < +\infty,$$

for some $\alpha \in (0,2)$. Then solutions $X(t)$, $Y(t)$, $t \geqslant 0$ are Hölder continuous with respect to $t > 0$ and $\theta \in T^d$ with any exponent smaller than $\frac{1}{2} - \frac{\alpha}{4}$.

The theorem is a consequence of Theorem 5.20 and Theorem 5.22 from [DaPrZa]. For the case of R^2 and the stochastic wave equation a similar result was obtained in [DaFr].

We finish the section with some applications of Theorems 3 and 4.

Corollary 1. Assume that $\Gamma \in L^2(T^d)$ and $d = 1, 2, 3$. Then the stochastic heat and wave equations (1.1) and (1.2) have solutions with values in $L^2(T^d)$.

Corollary 2. Assume that for some $1 \leqslant p \leqslant 2$, $\hat{\Gamma} \in l^p(\mathbb{Z}^d)$. If $d < \frac{2p}{p-1}$, then the stochastic heat and wave equations (1.1) and (1.2) have solutions in $L^2(T^d)$.

6 Conjectures

Taking into account Theorem 1 it is natural to expect that the following conjecture is true.

Conjecture 1. *If* $\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2}\mu(d\lambda) < +\infty$ *and functions* $g : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$ *are Lipschitz then nonlinear heat equation*

$$\begin{cases} \frac{\partial u(t,\theta)}{\partial t} = \Delta u(t,\theta) + g(u(t,\theta)) + b(u(t,\theta))\frac{\partial W_\Gamma}{\partial t}(t,\theta), \quad t > 0, \quad \theta \in \mathbb{R}^d \\ u(0,\theta) = 0, \quad \theta \in \mathbb{R}^d \end{cases} \tag{6.1}$$

and nonlinear wave equation

$$\begin{cases} \frac{\partial^2 u(t,\theta)}{\partial t^2} = \Delta u(t,\theta) + g(u(t,\theta)) + b(u(t,\theta))\frac{\partial W_\Gamma}{\partial t}(t,\theta), \quad t > 0, \quad \theta \in \mathbb{R}^d \\ u(0,\theta) = 0, \quad \frac{\partial u}{\partial t}(0,\theta) = 0, \quad \theta \in \mathbb{R}^d \end{cases} \tag{6.2}$$

have solutions.

At the moment there are only partial confirmations of the conjectures. Namely, the following result concerned with nonlinear heat equation (6.1) is contained in the paper [PeZa].

Theorem 6. *Assume that g and b are Lipschitz. Nonlinear heat equation (6.1) has a unique Markovian solution in* $L^2_{\rho_\kappa}$, *where* $\kappa > 0$, $\rho_\kappa(\theta) = e^{-\kappa|\theta|}$, $\theta \in \mathbb{R}^d$, *if either the spectral measure μ of W_Γ is finite or μ is infinite and has a density* $\frac{d\mu}{d\theta}$ *such that*:
i) if $d = 1$, $\frac{d\mu}{d\theta} \in L^p$ *for some* $p \in [1, +\infty]$;
ii) if $d = 2$, $\frac{d\mu}{d\theta} \in L^p$ *for some* $p \in [1, +\infty)$;
iii) if $d \geqslant 3$, $\frac{d\mu}{d\theta} \in L^p$ *for some* $p \in [1, \frac{d}{d-2})$.

A general existence result for nonlinear stochastic wave equations was proved in the paper [DaFr] by Dalang and Frangos. They showed the following result.

Theorem 7. *If correlation Γ is a function of the form $\Gamma(\theta) = f(|\theta|)$, with f positive and continuous outside 0, and if the dimension $d = 2$ and the condition (1.3) is satisfied then the equation (6.2) has a local solution.*

It is interesting to note that conditions of Theorem 5 imply that the reproducing kernel space S'_Γ consists only of functions, namely, $S'_\Gamma \subset L^2(\mathbb{R}^d) + C_b(\mathbb{R}^d)$, see [PeZa]. This seems to be essential for the definition of the stochastic integral with the integrands being multiplication operators. We therefore pose the following conjecture.

Conjecture 2. *If* $\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2}\mu(d\lambda) < +\infty$ *then elements of* S'_Γ *are represented by locally integrable functions.*

The following proposition is a partial confirmation of Conjecture 2.

Proposition 6. *If* $\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2}\mu(d\lambda) < +\infty$ *then* $\delta_{\{o\}} \notin S'_\Gamma$.

Proof: Assume, to the contrary, that for some $u \in L^2_{(s)}(\mathbb{R}^d, \mu)$, $\widehat{u\mu} = \delta_{\{o\}}$. Then, for a constant $c > 0$, $u(x)\mu(dx) = c\,dx$ and $u(x) > 0$ for almost all $x \in \mathbb{R}^d$ and measure μ can be identified with its density $\gamma(x) = \frac{c}{u(x)}$, $x \in \mathbb{R}^d$.
We also have

$$\int_{\mathbb{R}^d} \frac{1}{1+|\lambda|^2}\gamma(x)dx < +\infty$$

and

$$\int_{\mathbb{R}^d} u^2(x)\mu(dx) = \int_{\mathbb{R}^d} \frac{c^2}{\gamma^2(x)}\gamma(x)dx = \int_{\mathbb{R}^d} \frac{c^2}{\gamma(x)}dx < +\infty.$$

Consequently

$$\int_{\mathbb{R}^d} \frac{1}{\sqrt{1+|\lambda|^2}}\frac{\gamma(x)}{\sqrt{1+|\lambda|^2}}dx < +\infty,$$

$$\int_{\mathbb{R}^d} \frac{1}{\sqrt{1+|\lambda|^2}}\frac{\sqrt{1+|\lambda|^2}}{\gamma(x)}dx < +\infty.$$

Adding both inequalities and taking into account that $a + \frac{1}{a} \geqslant 2$ for all $a > 0$, one arrives at $\int_{\mathbb{R}^d} \frac{1}{\sqrt{1+|\lambda|^2}}d < +\infty$, a contradiction.

■

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